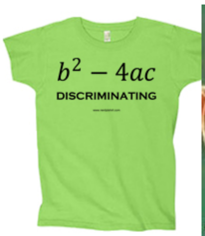
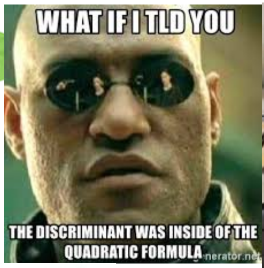
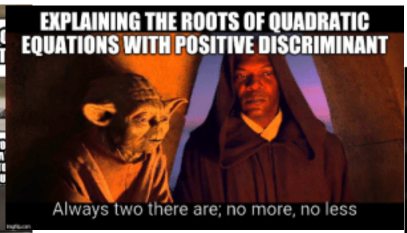
Discriminant:







oof

When $b^2 - 4ac < 0$:

Table of Contents

This is a long worksheet to cater for the students that want extra practice. If you want a shortcut, but still be sure to cover one of each type then follow the pink highlighted questions.

1 Bronze

1.1 Finding the number of roots

1) $f(x) = x^2 + 2x + 7$. Find the number of roots of the equation $f(x) = 0$.

2) $f(x) = x^2 + x + 3$. Find the number of roots of the equation $f(x) = 0$.

3) $f(x) = x^2 - 4x + 5$. Find the number of roots of the equation $f(x) = 0$.

1.2 Given number of roots, solve for an unknown

1.2.1 Equalities

4) The equation $kx^2 + (k-3)x + 1 = 0$ has two equal real roots. Find the possible values of k.

5) The equation $x^2 + 3kx + k = 0$, where p is a non-zero constant has equal roots. Find the possible values of k.

6) The equation $kx^2 + 12x + k = 0$, where k is a positive constant has two equal real roots. Find the possible values of k.

7) The equation $\frac{1}{2}x^2 + kx + 8 = 0$ has two equal roots. Find the possible values of k.

1.2.2 Inequalities

8) Find the range of values of k such that the equation $2x^2 + (3-k)x + k + 3 = 0$ has two distinct real solutions

9) Find the set of possible values of k such that the equation $x^2 + kx + (k+3) = 0$, where k is a constant, has different real roots.

10) The equation $kx^2 + 4x + (5-k) = 0$, where k is a constant, has two different real solutions for x. Find the possible values of k.

11) Find the set of possible values of k such that the equation $2x^2 - 3kx + (k+2) = 0$ has two distinct real roots.

12) The equation $x^2 + (k+2)x + 2k = 0$ has two distinct roots. Find the possible values of k.

2 Silver

2.1 Finding the number of roots

13) $3x + 4 = \frac{5}{x}$. Find the number of roots of the equation $f(x) = 0$.

2.2 Given number of roots, solve for an unknown

2.2.1 Equalities

14) The equation $x^2 + 2px + (3p + 4) = 0$, where p is a positive constant has equal roots.
    i. Find the value of p.
    ii. For this value of p, solve the equation $x^2 + 2px + (3p + 4) = 0$.

15) The quadratic equation $x + k + \frac{9}{x} = 0$ has equal roots. Find the two possible values of k.

2.2.2 Inequalities

16) The equation $x^2 - 3x + k^2 = 4$ has two distinct real solutions. Find the possible values of k.

17) The equation $x^2 + kx + 8 = k$ has no real solutions for x. Find the set of possible values of k.

18) The function f is defined by $f(x) = x^2 - 2x + k(3k + 2)$ where , $k \in \mathcal{R}$. Find the set of possible values for k for which $f(x) = 0$ has two distinct real roots.

2.3 Hidden Discriminant

2.3.1 Building your own equation

19) $y = k(2x^2 + 1)$ and $y = x^2 - 2x$, where k is a constant, touch each other. Determine the possible values of k.

20) The straight line with equation $y = 3x - 7$ does not cross or touch the curve with equation $y = 2px^2 - 6px + 4p$, where p is a constant. Show that $4p^2 - 20p + 9 < 0$ and find the possible values of p.

21) Use algebra to show that the curve $y = x(5 - x)$ and the line $2y = 5x + 4$ do not intersect.

22) A line has equation $kx + 2y - 3 = 0$, where k is a constant. Show that the line meets the curve $y = 3x^2 - 4x + 2$ only once when $k^2 - 16k + 40 = 0$ and hence find the exact two values of k for which the line is tangent to the curve.

23) Determine the exact values of k for which the curves $y = x^2 - kx$ and $y = 3(k+1) + kx - x^2$ touch.
    i. Determine whether or not there is a value of k for which the curves $y = x^2 - kx$ and $y = 3(k+1) + kx - x^2$ cross on the y axis.

24) Let $f(x) = m - \frac{1}{x}$, for $x \neq 0$. The line $y = x - m$ intersects the graph of f in two distinct points. Find the possible values of m.

25) Given that $y = -3x + c$, where c is a positive constant, meets the curve $y = \frac{1}{x} + 5$ at two distinct points. Show that $(5 - c)^2 > 12$. Hence find the range of possible values for c.

26)
    i. Sketch the curve with the equation $y = k - \frac{1}{2x}$ where k is a positive constant. State, in terms of k, The equation of the horizontal asymptote.

The straight line l has equation $y = 2x + 3$. Given that l cuts the curve in two distinct places,
    ii. Find the range of values of k, writing your answer in set notation

2.3.2 Tangent to curve

27) For what values of m is the line $y = mx + 5$ a tangent to the parabola $y = 4 - x^2$

28) Show that the line with equation $y = 2x - 3$ is tangent to the curve with equation $y = x^2 - 2$.

29) Prove that the line $y = 4x - 9$ is tangent to the curve with equation $y = 4x(x - 2)$.

30) Find the values of the constant k for which the straight line $y = 2x + k$ is a tangent to the curve $y = x^2 - 4x + 2$.

31) The line $y = 4x + c$, where c is a constant, is a tangent to the curve with equation $y = 2x^2 + 8x + 3$. Calculate the value of c.

32) The line $y = 3x + k$, where k is a positive constant, is a tangent to the curve with equation $y = \frac{1}{3}x^2 + 8$. Find the value of k.

33) For what values of m is the line $y = mx + 5$ a tangent to the parabola $y = 4 - x^2$?

34) The line $y = 5x - 3$ is tangent to the curve $y = kx^2 - 3x + 5$ at the point A.
    i. Find the value of k.
    ii. Find the coordinates of A.

35) Find the values of the constant c for which the straight line $y = c - 3x$ is a tangent to the curve $y = \frac{3}{x}$.

36) The line $y = 3x + k$ is a tangent to the curve $x^2 + xy + 16 = 0$.
    i. Find the possible values of k.
    ii. For each of these values of k, find the coordinates of the point of contact of the tangent with the curve.

37) Consider the two curves with equations $y = x^2 + kx + k$, where $k < 0$ is a constant and $y = -x^2 + 2x - 4$. Both curves pass through the point P and the tangent to P to one of the curves is also a tangent to P to the other curve.
    i. Find the value of k.
    ii. Find the coordinates of P.

38) The line $y = m(x - m)$ is a tangent to the curve $(1 - x)y = 1$. Determine m and the coordinates of the point where the tangent meets the curve.
Hint: Need calculator to solve this.

39) A curve has equation $y = x^2 - 4x + 4$ and a line has equation $y = mx$, where m is a constant.

i. For the case where $m = 1$, the curve and the line intersect at the points A and B. Find the coordinates of the midpoint of AB.
ii. Find the non-zero value of m for which the line is tangent to the curve and find the coordinates of the point where the tangent touches the curve.

3 Gold

3.1 Finding the number of roots

40) $f(x) = x^2 - kx - 5, k \neq 0$. Find the number of roots of the equation $f(x) = 0$.

41) $f(x) = px^2 + qx - 4p, p \neq 0$. Find the number of roots of the equation $f(x) = 0$.

3.2 Given number of roots, solve for an unknown

3.2.1 Equalities

42) Let $f(x) = px^2 + (10 - p)x + \frac{5}{4}p - 5$. Find the values of p such that $f(x) = 0$ has two equal roots.

3.2.2 Inequalities

43) The quadratic function Q is defined by $Q(x) = kx^2 - (k - 3)x + (k - 8)$. Determine the values of k for which $Q(x) = 0$ has no real roots.

44) The equation $(k + 3)x^2 + 6x + k = 5$, where k is a constant has 2 distinct real solutions for x. Find the set of possible values of k.

45) The quadratic equation $(2k - 7)x^2 - (k - 2)x + (k - 3) = 0$ has **real roots.**
    i. Show that $7k^2 - 48k + 80 \leq 0$.
    ii. Find the possible values of k.

46) The equation $k(3x^2 + 8x + 9) = 2 - 6x$ where k is a real constant, has no real roots.
    i. Show that k satisfies the inequality $11k^2 - 30k - 9 > 0$.
    ii. Find the range of possible values of k.

3.3 Given number of roots, showing existence for all real values

47) Show that the equation $x^2 - (5 - k)x - (k + 2) = 0$ has two distinct real roots for all values of k.

48) $f(x) = x^2 + (k + 3)x + k$, where k is a real constant.
    i. Show that the discriminant of $f(x)$ can be expressed in the form $(k + a)^2 + b$, where a and b are integers to be found.
    ii. Show that for all values of k, the equation $f(x) = 0$ has real roots.

Note: We can't show discriminant equal zero here but that doesn't matter since for real roots we need either $b^2 - 4ac > 0$ or $b^2 - 4ac \geq 0$. It's enough to show just one these ie the > case.

49) $h(x) = 2x^2 + (k+4)x + k$. Show that $h(x)$ *has* two distinct real roots for all values of k.

50) Prove that the quadratic equation $(q-5)x^2 + 5x - q = 0$ has real roots for any values of q.

51) $f(x) = 4kx^2 + (4k+2)x + 1$
   i. Prove that $f(x)$ *has* two distinct real roots for all non-zero values of k.
   ii. Explain why $f(x)$ cannot have 2 distinct real roots when $k = 0$.
   Hint: when $k = 0$, $f(x) = 2x + 1$ which is a linear function with one root so cannot have 2 distinct real roots

3.4 Hidden Discriminant

3.4.1 Tangent to a circle

52) A particular circle has the equation $x^2 + y^2 + 4x - 10y - 7 = 0$. A line has equation of the form $y = 2x + c$.
   i. Show that where the line meets the circle $5x^2 + 4(c-4)x + (c^2 - 10c - 7) = 0$.
   ii. Hence show that the line will be tangent if $c^2 - 18c - 99 = 0$.
   iii. Hence find the two values of c for which the line is tangent to the circle.

53) The line with equation $mx - y - 2 = 0$ touches the circle with equation $x^2 + 6x + y^2 - 8y = 4$. Find the two possible values of m, giving your answer in exact form.

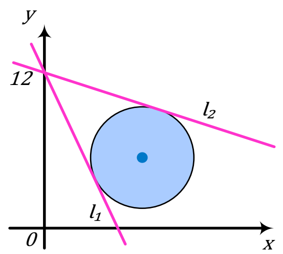
54) The line with equation $2x + y - 5 = 0$ is a tangent to the circle with equation $(x-3)^2 + (y-p)^2 = 5$.
   i. Find the 2 possible values of p.
   ii. Write down the coordinates of the centre of the circle in each case.

55) The circle C has equation $(x-7)^2 + (y+1)^2 = 5$. The line l with positive gradient passes through $(0,-2)$ and is tangent to the circle. Find the equation of l, giving your answer in the form $y = mx + c$.

56) The circle C has equation $(x+5)^2 + (y+3)^2 = 80$. The line l is a tangent to the circle and has gradient 2. Find two possible equations for l giving your answers in the form $y = mx + c$.

57) The circle C has equation $(x-5)^2 + (y+3)^2 = 10$. The line l is a tangent to the circle and had gradient -3. Find two possible equations for l, giving your answers in the form $y = mx + c$.

58) The circle C has equation $(x-6)^2 + (y-5)^2 = 17$.



The lines l_1 and l_2 are tangent to the circle and intersect at the point (0,12). Find the equations of l_1 and l_2, giving your answer in the form $y = mx + c$.
Hint: Substitute $y = mx + 12$ into circle equation.

3.4.2 Showing always positive or negative

59) Prove that, for all values of x, $x^2 + 6x + 18 > 2 - \frac{1}{2}x$

60) Consider the function $f(x) = (x-1)^2$. Show that $f(x) \geq 0$ for all real values of x.

$$f(x) > 0$$
$$(x-1)^2 > 0$$
$$x^2 - 2x + 1 > 0$$

61) Consider the function $f(x) = x^2 + 16$. Show that $f(x) > 0$ for all real values of x.

62) Consider the function $f(x) = x^2 - 8x + 18$. Show that $f(x) > 0$ for all real values of x.

63) Show that the function $f(x) = 2x^2 + 8x + 9$ is either always positive **or** always negative for any value of x.
Hint: Show discriminant is always less than zero.

64) Consider the function $f(x) = -5x^2 - x - 10$. Show that $f(x) < 0$ for all real values of x.

4 Diamond

4.1 Given number of roots, solve for an unknown

4.1.1 Equalities

65) The equation $2kx^2 - 4kx + 1 = 0$, for $k \neq 0$ has two equal real solutions.
  i. Find the possible values of k.
  ii. The line $y = p$ intersects the graph of f. Find all possible values of p.

4.1.2 Inequalities

66) Find the values for m such that the equation $mx^2 - 2(m+2)x + m + 2 = 0$ has
  i. Two real roots
  ii. Two real roots, one positive and one negative

67) A quadratic curve has the equation $f(x) = 2x^2 + (4k+3)x + (2k-1)(k+2), x \in \mathbb{R}$ where k is a constant.
  i. Evaluate the discriminant of $f(x)$.
  ii. Express $f(x)$ as a product of 2 linear factors.

4.2 Given number of roots, Showing existence for all real values

68) The equation $px^2 + qx + r = 0$, where p, q, and r are constants, has roots $-\frac{1}{2}$ *and* $\frac{3}{4}$. Find the smallest possible integer values of p, q and r.

69)
  i. Prove that the equation $3x^2 + 2kx + k - 1 = 0$ has two distinct real roots for all values of $k \in \mathcal{R}$.
  ii. Find the value of k for which the two roots of the equation are closest together.

4.3 Hidden Discriminant - Combined With Other Topics

4.3.1 Showing Always Positive Or Always Negative

70) Show that the function $f(x) = \frac{1}{3}x^3 - 4x^2 + 18x$ is always increasing.

71) Let $f(x) = px^3 + px^2 + qx$. Given that $f'(x) \geq 0$, show that $p^2 \leq 3pq$.

72) Find the range of the values of m such that for all x, $m(x+1) \leq x^2$.

73) Consider the function $f(x) = (2p-3)x^2 + 2x + p - 1, p \in \Re$. Find the set of values of p such that $f(x) > 0$.

4.3.2 Combined With Other Topics

4.3.2.1 Differentiation

74) Given that the graph of $y = x^3 - 6x^2 + kx - 4$ has exactly one point at which the gradient is zero, find the value of k.

75) The cubic curve $y = 8x^3 + bx^2 + cx + d$ has 2 distinct points P and Q, where the gradient is zero.

i. Show that $b^2 > 24c$.
Hint: Differentiate and solve discriminant > 0.

ii. Given that the coordinates of P and Q are $\left(\frac{1}{2}, -12\right)$ and $\left(-\frac{3}{2}, 20\right)$ respectively, find the values of b, c and d.

76) Show that the curve $y = \frac{x^2-1}{x+a}$ has no turning points if $-1 < a < 1$.

77) The function f is given by $f(x) = e^{mx}(x^2 + x)$, $x \in \mathbb{R}$, where m is a non-zero constant.
Show that f has two stationary points, for all non-zero values of m.

4.3.2.2 Logs

78) Consider $f(x) = \log_k(6x - 3x^2)$, for $0 < x < 2$, where $k > 0$. The equation $f(x) = 2$, has exactly one solution. Find the value of k.

4.3.2.3 Trig

79) Let $f(x) = 3\tan^4 x + 2k$ *and* $g(x) = -\tan^4 x + 8k\tan^2 x + k$, for $0 \le x \le 1$, *where* $0 < k < 1$. The graphs of f and g intersect at exactly one point. Find the value of k.
Hint: This is a quartic but you do the same thing. Just use a substitution and your coefficients remain the same.

4.3.2.4 Quadratic Modelling

80) A person throws a ball in a sports hall. The height of the ball, h m can be modelled in relation to the horizontal distance from the point it was thrown from by the quadratic equation.

$$h(x) = -\frac{3}{10}x^2 + \frac{5}{2}x + \frac{3}{2}$$

The ball has a sloping ceiling which can be modelled with equation $h(x) = \frac{15}{2} - \frac{1}{5}x$.
Determine whether the model predicts that the ball will hit the ceiling.
Hint: Represent hitting i.e. solve simultaneously and use discriminant to see how many solutions.

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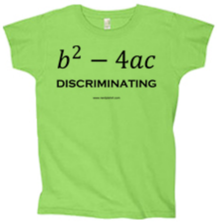
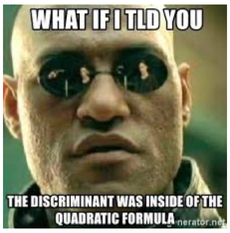
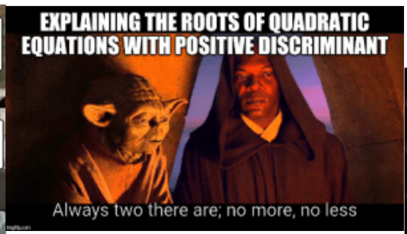
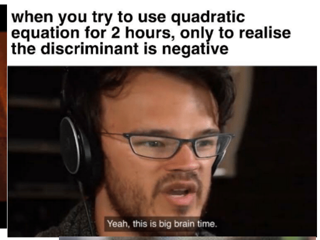
Discriminant:









oof

Table of Contents

1 Bronze

1.1 Finding the number of roots

1)

$$x^2 + 2x + 7 = 0$$

When the questions talks about "number of roots" or "number of solutions" we must use the discriminant. We aren't finding the actual solution, we are just talking about "how many" and this is our hint to use the solution.

Step 1: Make sure we have 0 on one side. We already have this.

Step 2: Locate $a, b,$and c

$$1x^2 + 2x + 7 = 0$$

a =number in front of $x^2 = 1$
b =number in front of x $= 2$
c =number without an x or x^2 =7

Step 3: Plug into the formula $b^2 - 4ac$

$$2^2 - 4(1)(7)$$

Step 4: Simplify

$$4 - 28 = -24$$

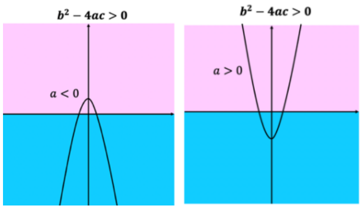
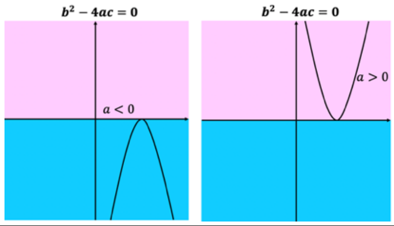
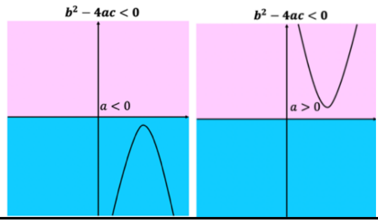
Step 5: Find the discriminant and decide which case you're in

Case 1: $b^2 - 4ac > 0$ (2 real distinct/different roots)
Case 2: $b^2 - 4ac = 0$ (2 real repeated/double roots i.e. same/equal roots)
Case 3: $b^2 - 4ac < 0$ (no real roots)
We can combine case 1 and 3 if the question just says real and doesn't mention whether distinct of the repeated
Case 4: $b^2 - 4ac \geq 0$ (2 real roots i.e. doesn't mention whether distinct or repeated)

$-24 < 0$ so we are in case 3 which means no real roots

You should just memorise these cases, but let's see graphically why they make sense. The discriminant is just the part underneath the square root of the quadratic formula

Case 1	Case 2	Case 3
$x = \frac{-b \pm \sqrt{positive}}{2a}$ will give 2 solutions $x = \frac{-b + something}{2a}$ $x = \frac{-b - something}{2a}$	$x = \frac{-b \pm \sqrt{zero}}{2a}$ will give 1 solution $x = \frac{-b \pm 0}{2a} = -\frac{b}{2a}$	$x = \frac{-b \pm \sqrt{negative}}{2a}$ will give no solutions. $x = \frac{-b \pm undefined\ value}{2a}$

i.e. 2 places where the graph crosses the x axis 	i.e. 1 place where the graph crosses the x axis 	i.e. 0 places where the graph crosses the x axis 

2)

$$x^2+x+3=0$$

When the questions talks about "number of roots" or "number of solutions" we must use the discriminant. We aren't finding the actual solution, we are just talking about "how many" and this is our hint to use the solution.

Step 1: Make sure we have 0 on one side. We already have this.

Step 2: Locate a, b,and c

$$1x^2+1x+3=0$$

a =number in front of x^2 $=1$
b =number in front of x $=1$
c =number without an x or x^2 $=3$

Step 3: Plug into the formula b^2-4ac

$$1^2-4(1)(3)$$

Step 4: Simplify

$$4-12=-11$$

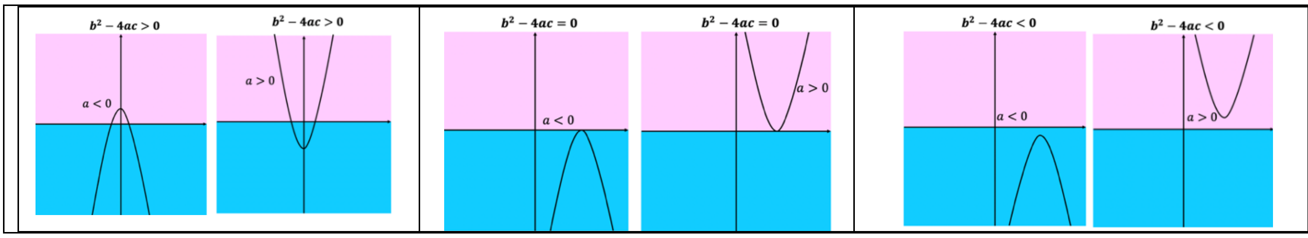
Step 5: Find the discriminant and decide which case you're in

Case 1: $b^2-4ac>0$ (2 real distinct/different roots)
Case 2: $b^2-4ac=0$ (2 real repeated/double roots i.e. same/equal roots)
Case 3: $b^2-4ac<0$ (no real roots)
We can combine case 1 and 3 if the question just says real and doesn't mention whether distinct of the repeated
Case 4: $b^2-4ac\geq 0$ (2 real roots i.e. doesn't mention whether distinct or repeated)

$-11<0$ so we are in case 3 which means no real roots

You should just memorise these cases, but let's see graphically why they make sense. The discriminant is just the part underneath the square root of the quadratic formula

Case 1	**Case 2**	**Case 3**
$x=\frac{-b\pm\sqrt{positive}}{2a}$ will give 2 solutions $x=\frac{-b+something}{2a}$ $x=\frac{-b-something}{2a}$ i.e. 2 places where the graph crosses the x axis	$x=\frac{-b\pm\sqrt{zero}}{2a}$ will give 1 solution $x=\frac{-b\pm0}{2a}=-\frac{b}{2a}$ i.e. 1 place where the graph crosses the x axis	$x=\frac{-b\pm\sqrt{negative}}{2a}$ will give no solutions. $x=\frac{-b\pm undefined\ value}{2a}$ i.e. 0 places where the graph crosses the x axis



3)

$$x^2 - 4x + 5 = 0$$

When the questions talks about "number of roots" or "number of solutions" we must use the discriminant. We aren't finding the actual solution, we are just talking about "how many" and this is our hint to use the solution.

Step 1: Make sure we have 0 on one side. We already have this.

Step 2: Locate a, b,and c

$$1x^2 - 4x + 5 = 0$$

a =number in front of $x^2 = 1$
b =number in front of $x = -4$
c =number without an x or $x^2 = 5$

Step 3: Plug into the formula $b^2 - 4ac$

$$(-4)^2 - 4(1)(5)$$

Step 4: Simplify

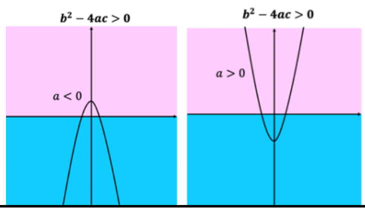
$$16 - 20 = -4$$

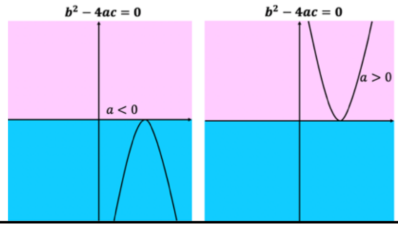
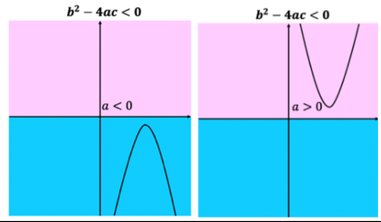
Step 5: Find the discriminant and decide which case you're in

Case 1: $b^2 - 4ac > 0$ (2 real distinct/different roots)
Case 2: $b^2 - 4ac = 0$ (2 real repeated/double roots i.e. same/equal roots)
Case 3: $b^2 - 4ac < 0$ (no real roots)
We can combine case 1 and 3 if the question just says real and doesn't mention whether distinct of the repeated
Case 4: $b^2 - 4ac \geq 0$ (2 real roots i.e. doesn't mention whether distinct or repeated)

$-4 < 0$ so we are in case 3 which means no real roots

You should just memorise these cases, but let's see graphically why they make sense. The discriminant is just the part underneath the square root of the quadratic formula

Case 1	Case 2	Case 3
$x = \frac{-b \pm \sqrt{positive}}{2a}$ will give 2 solutions $x = \frac{-b + something}{2a}$ $x = \frac{-b - something}{2a}$ i.e. 2 places where the graph crosses the x axis 	$x = \frac{-b \pm \sqrt{zero}}{2a}$ will give 1 solution $x = \frac{-b \pm 0}{2a} = -\frac{b}{2a}$ i.e. 1 place where the graph crosses the x axis	$x = \frac{-b \pm \sqrt{negative}}{2a}$ will give no solutions. $x = \frac{-b \pm undefined\ value}{2a}$ i.e. 0 places where the graph crosses the x axis

1.2 Given number of roots, solve for an unknown

1.2.1 Equalities

4)

$$kx^2 + (k-3)x + 1 = 0$$

When the questions talks about "number of roots" or "number of solutions" we must use the discriminant. We are finding the actual solution, we are just talking about "how many" and this is our hint to use the solution.

Step 1: Make sure we have 0 on one side. We already have this.

Step 2: locate $a, b,$and c

$$kx^2 + (k-3)x + 1 = 0$$

a =number in front of $x^2 = k$
b =number in front of $x = k - 3$
c =number without an x or $x^2 = 1$

Note: Don't let the fact that we have a k confuse you. k is considered a constant here. x is the variable of the equation and that is the non-constant i.e. the variable that we don't include in a, b or c

$$kx^2 + (k-3)x + 1 = 0$$

Step 3: We are given what case we are in, so we plug into the correct formula for that case
Case 1: $b^2 - 4ac > 0$ (2 real distinct/different roots)
Case 2: $b^2 - 4ac = 0$ (2 real repeated/double roots i.e. same/equal roots)
Case 3: $b^2 - 4ac < 0$ (no real roots)
Case 4: $b^2 - 4ac \geq 0$ (2 real roots i.e. doesn't mention whether distinct or repeated)

We are told 2 equal roots, hence we are in case 2:

$$(k-3)^2 - 4(k)(1) = 0$$

Step 4: Now we have an equation to solve to find k

$$(k-3)^2 - 4k(1) = 0$$
$$k^2 - 6k + 9 - 4k = 0$$
$$k^2 - 10k + 9 = 0$$
$$(k-9)(k-1) = 0$$
$$k = 9, k = 1$$

5)

Here we are now given the case that we are in and have to find an unknown. We work backward from building an equation based on knowing what the discriminant is equal to

Step 1: Make sure we have 0 on one side. We already have this.

Step 2: locate $a, b,$and c

$$x^2 + 3kx + k = 0$$

a =number in front of $x^2 = 1$

b =number in front of $x = 3k$
c =number without an x or $x^2 = k$

Note: Don't let the fact that we have a k confuse you. k is considered a constant here. x is the variable of the equation and that is the non-constant i.e. the variable that we don't include in a, b or c

$$x^2 + 3kx + k = 0$$

Step 3: We are given what case we are in, so we plug into the correct formula for that case

Case 1: $b^2 - 4ac > 0$ (2 real distinct/different roots)
Case 2: $b^2 - 4ac = 0$ (2 real repeated/double roots i.e. same/equal roots)
Case 3: $b^2 - 4ac < 0$ (no real roots)
Case 4: $b^2 - 4ac \geq 0$ (2 real roots i.e. doesn't mention whether distinct or repeated)

We are told 2 equal roots, hence we are in case 2:

$$(3k)^2 - 4(1)(k) = 0$$

Step 4: Now we have an equation to solve to find k

$$9k^2 - 4k = 0$$
$$k(9k - 4) = 0$$
$$k = 0, k = \frac{4}{9}$$

6)

Here we are now given the case that we are in and have to find an unknown. We work backward from building an equation based on knowing what the discriminant is equal to

Step 1: Make sure we have 0 on one side. We already have this.

Step 2: locate $a, b,$and c

$$kx^2 + 12x + k = 0$$

a =number in front of $x^2 = k$
b =number in front of $x = 12$
c =number without an x or $x^2 = k$

Note: Don't let the fact that we have a k confuse you. k is considered a constant here. x is the variable of the equation and that is the non-constant i.e. the variable that we don't include in a, b or c

$$kx^2 + 12x + k = 0$$

Step 3: We are given what case we are in, so we plug into the correct formula for that case

Case 1: $b^2 - 4ac > 0$ (2 real distinct/different roots)
Case 2: $b^2 - 4ac = 0$ (2 real repeated/double roots i.e. same/equal roots)
Case 3: $b^2 - 4ac < 0$ (no real roots)
Case 4: $b^2 - 4ac \geq 0$ (2 real roots i.e. doesn't mention whether distinct or repeated)

We are told 2 equal roots, hence we are in case 2:

$$(12)^2 - 4(k)(k) = 0$$

Step 4: Now we have an equation to solve to find k

$$144 - 4k^2 = 0$$
$$k^2 = 36$$
$$k = \pm 6$$

7)

Here we are now given the case that we are in and have to find an unknown. We work backward from building an equation based on knowing what the discriminant is equal to

Step 1: Make sure we have 0 on one side. We already have this.

Step 2: locate $a, b,$and c

$$\frac{1}{2}x^2 + kx + 8 = 0$$

a =number in front of $x^2 = \frac{1}{2}$
b =number in front of $x = k$
c =number without an x or $x^2 = 8$

Note: Don't let the fact that we have a k confuse you. k is considered a constant here. x is the variable of the equation and that is the non-constant i.e. the variable that we don't include in a, b or c

$$\frac{1}{2}x^2 + kx + 8 = 0$$

Step 3: We are given what case we are in, so we plug into the correct formula for that case
Case 1: $b^2 - 4ac > 0$ (2 real distinct/different roots)
Case 2: $b^2 - 4ac = 0$ (2 real repeated/double roots i.e. same/equal roots)
Case 3: $b^2 - 4ac < 0$ (no real roots)
Case 4: $b^2 - 4ac \geq 0$ (2 real roots i.e. doesn't mention whether distinct or repeated)

We are told 2 equal roots, hence we are in case 2:

$$(k)^2 - 4\left(\frac{1}{2}\right)(8) = 0$$

Step 4: Now we have an equation to solve to find k

$$k^2 - 16 = 0$$
$$k^2 = 16$$
$$k = \pm 4$$

1.2.2 Inequalities

8)

$$2x^2 + (3 - k)x + k + 3 = 0$$

When the questions talks about "number of roots" or "number of solutions" we must use the discriminant. We are finding the actual solution, we are just talking about "how many" and this is our hint to use the solution.

Step 1: Make sure we have 0 on one side. We already have this.

Step 2: locate $a, b,$and c

$$2x^2 + (3 - k)x + k + 3 = 0$$

a =number in front of $x^2 = 2$
b =number in front of $x = 3 - k$
c =number without an x or $x^2 = k + 3$

Note: Don't let the fact that we have a k confuse you. k is considered a constant here. x is the variable of the equation and that is the non-constant i.e. the variable that we don't include in a, b or c

$$2x^2 + (3 - k)x + k + 3 = 0$$

Step 3: We are given what case we are in, so we plug into the correct formula for that case
Case 1: $b^2 - 4ac > 0$ (2 real distinct/different roots)
Case 2: $b^2 - 4ac = 0$ (2 real repeated/double roots i.e. same/equal roots)
Case 3: $b^2 - 4ac < 0$ (no real roots)

Case 4: $b^2 - 4ac \geq 0$ (2 real roots i.e. doesn't mention whether distinct or repeated)

We are told 2 two distinct real solutions hence we are in case 1

$$(3-k)^2 - 4(2)(k+3) > 0$$

Step 4: Now we have an inequality to solve to find k

$$(3-k)^2 - 4(2)(k+3) > 0$$
$$9 - 6k + k^2 - 8k - 24 > 0$$
$$k^2 - 14k - 15 > 0$$
$$(k-15)(k+1) > 0$$
$$k = 15, k = -1$$

Watch out for a really common mistake: Do not just guess the signs when solving a quadratic inequality, make sure to either graph or use a sign change test with a number line

$$k > 15, k < -1$$

9)

$$x^2 + kx + (k+3) = 0$$

When the questions talks about "number of roots" or "number of solutions" we must use the discriminant. We are finding the actual solution, we are just talking about "how many" and this is our hint to use the solution.

Step 1: Make sure we have 0 on one side. We already have this.

Step 2: locate a, b,and c

$$1x^2 + kx + (k+3) = 0$$

a =number in front of $x^2 = 1$
b =number in front of $x = k$
c =number without an x or $x^2 = k + 3$

Note: Don't let the fact that we have a k confuse you. k is considered a constant here. x is the variable of the equation and that is the non-constant i.e. the variable that we don't include in a, b or c

$$x^2 + kx + k + 3 = 0$$

Step 3: We are given what case we are in, so we plug into the correct formula for that case

Case 1: $b^2 - 4ac > 0$ (2 real distinct/different roots)
Case 2: $b^2 - 4ac = 0$ (2 real repeated/double roots i.e. same/equal roots)
Case 3: $b^2 - 4ac < 0$ (no real roots)
Case 4: $b^2 - 4ac \geq 0$ (2 real roots i.e. doesn't mention whether distinct or repeated)

We are told 2 two distinct real solutions hence we are in case 1

$$k^2 - 4(1)(k+3) > 0$$

Step 4: Now we have an inequality to solve to find k

$$k^2 - 4(1)(k+3) > 0$$
$$k^2 - 4k - 12 > 0$$
$$(k-6)(k+2) > 0$$
$$k = 6, k = -2$$

Watch out for a really common mistake: Do not just guess the signs when solving a quadratic inequality, make sure to either graph or use a sign change test with a number line

$$k > 6, k < -2$$

10)

$$kx^2 + 4x + (5 - k) = 0$$

When the questions talks about "number of roots" or "number of solutions" we must use the discriminant. We are finding the actual solution, we are just talking about "how many" and this is our hint to use the solution.

Step 1: Make sure we have 0 on one side. We already have this.

Step 2: locate a, b,and c

$$kx^2 + 4x + (5 - k) = 0$$

a =number in front of $x^2 = k$
b =number in front of $x = 4$
c =number without an x or $x^2 = 5 - k$

Note: Don't let the fact that we have a k confuse you. k is considered a constant here. x is the variable of the equation and that is the non-constant i.e. the variable that we don't include in a, b or c

$$kx^2 + 4x + 5 - k = 0$$

Step 3: We are given what case we are in, so we plug into the correct formula for that case
Case 1: $b^2 - 4ac > 0$ (2 real distinct/different roots)
Case 2: $b^2 - 4ac = 0$ (2 real repeated/double roots i.e. same/equal roots)
Case 3: $b^2 - 4ac < 0$ (no real roots)
Case 4: $b^2 - 4ac \geq 0$ (2 real roots i.e. doesn't mention whether distinct or repeated)

We are told 2 two distinct real solutions hence we are in case 1

$$4^2 - 4(k)(5 - k) > 0$$

Step 4: Now we have an inequality to solve to find k

$$4^2 - 4(k)(5 - k) > 0$$
$$16 - 20k + 4k^2 > 0$$
$$4k^2 - 20k + 16 > 0$$
$$k^2 - 5k + 4 > 0$$
$$(k - 4)(k - 1) > 0$$
$$k = 4, k = 1$$

Watch out for a really common mistake: Do not just guess the signs when solving a quadratic inequality, make sure to either graph or use a sign change test with a number line

$$k < 1, k > 4$$

11)

$$2x^2 - 3kx + (k + 2) = 0$$

When the questions talks about "number of roots" or "number of solutions" we must use the discriminant. We are finding the actual solution, we are just talking about "how many" and this is our hint to use the solution.

Step 1: Make sure we have 0 on one side. We already have this.

Step 2: locate a, b,and c

$$2x^2 - 3kx + (k + 2) = 0$$

a =number in front of $x^2 = 2$
b =number in front of $x = -3k$
c =number without an x or $x^2 = k + 2$

Note: Don't let the fact that we have a k confuse you. k is considered a constant here. x is the variable of the equation and that is the non-constant i.e. the variable that we don't include in a, b or c

$$2x^2 + 3kx + k + 2 = 0$$

Step 3: We are given what case we are in, so we plug into the correct formula for that case

Case 1: $b^2 - 4ac > 0$ (2 real distinct/different roots)
Case 2: $b^2 - 4ac = 0$ (2 real repeated/double roots i.e. same/equal roots)
Case 3: $b^2 - 4ac < 0$ (no real roots)
Case 4: $b^2 - 4ac \geq 0$ (2 real roots i.e. doesn't mention whether distinct or repeated)

We are told 2 two distinct real solutions hence we are in case 1

$$(-3k)^2 - 4(2)(k+2) > 0$$

Step 4: Now we have an inequality to solve to find k

$$(-3k)^2 - 4(2)(k+2) > 0$$
$$9k^2 - 8k - 16 > 0$$
Assume $9k^2 - 8k - 16 = 0.$
$$k = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(9)(-16)}}{2(9)} = \frac{8 \pm \sqrt{640}}{18} = \frac{8 \pm 8\sqrt{10}}{18} = \frac{4 \pm 4\sqrt{10}}{9}$$
For $9k^2 - 8k - 16 > 0$,

Watch out for a really common mistake: Do not just guess the signs when solving a quadratic inequality, make sure to either graph or use a sign change test with a number line

For $9k^2 - 8k - 16 > 0$,
$$k > \frac{4 + 4\sqrt{10}}{9}, k < \frac{4 - 4\sqrt{10}}{9}$$

12)

$$x^2 + (k+2)x + 2k = 0$$

When the questions talks about "number of roots" or "number of solutions" we must use the discriminant. We are finding the actual solution, we are just talking about "how many" and this is our hint to use the solution.

Step 1: Make sure we have 0 on one side. We already have this.

Step 2: locate a, b,and c

$$x^2 + (k+2)x + 2k = 0$$

a =number in front of $x^2 = 1$
b =number in front of $x = k + 2$
c =number without an x or $x^2 = 2k$

Note: Don't let the fact that we have a k confuse you. k is considered a constant here. x is the variable of the equation and that is the non-constant i.e. the variable that we don't include in a, b or c

$$x^2 + (k+2)x + 2k = 0$$

Step 3: We are given what case we are in, so we plug into the correct formula for that case

Case 1: $b^2 - 4ac > 0$ (2 real distinct/different roots)
Case 2: $b^2 - 4ac = 0$ (2 real repeated/double roots i.e. same/equal roots)
Case 3: $b^2 - 4ac < 0$ (no real roots)
Case 4: $b^2 - 4ac \geq 0$ (2 real roots i.e. doesn't mention whether distinct or repeated)

We are told 2 two distinct real solutions hence we are in case 1

$$(k+2)^2 - 4(1)(2k) > 0$$

Step 4: Now we have an inequality to solve to find k

$$(k-2)^2 > 0$$
$$k = 2$$

Watch out for a really common mistake: Do not just guess the signs when solving a quadratic inequality, make sure to either graph or use a sign change test with a number line

$$k > 4$$

2 Silver

2.1 Finding the number of roots

13)

$$3x^2 + 4x = 5$$

When the questions talks about "number of roots" or "number of solutions" we must use the discriminant. We aren't finding the actual solution, we are just talking about "how many" and this is our hint to use the solution.

Step 1: Make sure we have 0 on one side.

$$3x^2 + 4x - 5 = 0$$

Step 2: Locate a, b,and c

$$3x^2 + 4x - 5 = 0$$

a =number in front of $x^2 = 3$
b =number in front of $x = 4$
c =number without an x or $x^2 = -5$

Step 3: Plug into the formula $b^2 - 4ac$

$$(4)^2 - 4(3)(-5)$$

Step 4: Simplify

$$16 + 60 = 76$$

Step 5: Find the discriminant and decide which case you're in

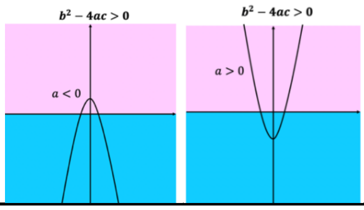
Case 1: $b^2 - 4ac > 0$ (2 real distinct/different roots)
Case 2: $b^2 - 4ac = 0$ (2 real repeated/double roots i.e. same/equal roots)
Case 3: $b^2 - 4ac < 0$ (no real roots)
We can combine case 1 and 3 if the question just says real and doesn't mention whether distinct of the repeated
Case 4: $b^2 - 4ac \geq 0$ (2 real roots i.e. doesn't mention whether distinct or repeated)

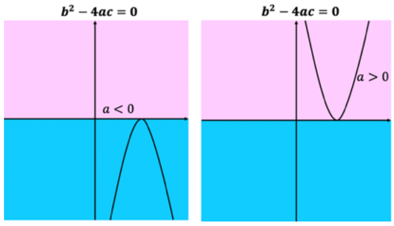
$76 > 0$ so we are in case 1 which means 2 real distinct roots

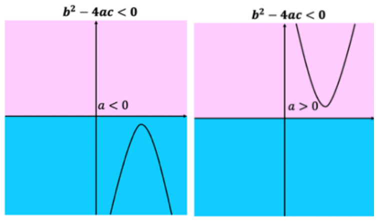
You should just memorise these cases, but let's see graphically why they make sense. The discriminant is just the part underneath the square root of the quadratic formula

Case 1	Case 2	Case 3
$x = \frac{-b \pm \sqrt{positive}}{2a}$ will give 2 solutions	$x = \frac{-b \pm \sqrt{zero}}{2a}$ will give 1 solution	$x = \frac{-b \pm \sqrt{negative}}{2a}$ will give no solutions.

$x = \frac{-b+something}{2a}$ $x = \frac{-b-something}{2a}$ i.e. 2 places where the graph crosses the x axis	$x = \frac{-b\pm 0}{2a} = -\frac{b}{2a}$ i.e. 1 place where the graph crosses the x axis	$x = \frac{-b\pm undefined\ value}{2a}$ i.e. 0 places where the graph crosses the x axis







2.2 Given number of roots, solve for an unknown

2.2.1 Equalities

14)

$$x^2 + 2px + (3p + 4) = 0$$

i.

When the questions talks about "number of roots" or "number of solutions" we must use the discriminant. We are finding the actual solution, we are just talking about "how many" and this is our hint to use the solution.

Step 1: Make sure we have 0 on one side. We already have this.

Step 2: locate a, b,and c

$$1x^2 + 2px + (3p + 4) = 0$$

a =number in front of $x^2 = 1$
b =number in front of $x = 2p$
c =number without an x or $x^2 = 3p + 4$

Note: Don't let the fact that we have a p confuse you. p is considered a constant here. x is the variable of the equation and that is the non-constant i.e. the variable that we don't include in a, b or c

$$x^2 + 2px + 3p + 4 = 0$$

Step 3: We are given what case we are in, so we plug into the correct formula for that case
Case 1: $b^2 - 4ac > 0$ (2 real distinct/different roots)
Case 2: $b^2 - 4ac = 0$ (2 real repeated/double roots i.e. same/equal roots)
Case 3: $b^2 - 4ac < 0$ (no real roots)
Case 4: $b^2 - 4ac \geq 0$ (2 real roots i.e. doesn't mention whether distinct or repeated)

We are told 2 equal roots, hence we are in case 2:

$$(2p)^2 - 4(1)(3p + 4) = 0$$

Step 4: Now we have an equation to solve to find p

$$(2p)^2 - 4(1)(3p + 4) = 0$$
$$4p^2 - 12p - 16 = 0$$
$$p^2 - 3p - 4 = 0$$

$$(p - 4)(p + 1) = 0$$
$$p = 4, p = -1$$

Since p is positive, $p = 4$.

ii.

Substitute $p = 4$ into $x^2 + 2px + (3p + 4) = 0$,

$$x^2 + 2(4)x + (3(4) + 4) = 0$$
$$x^2 + 8x + 16 = 0$$
$$(x + 4)^2 = 0$$
$$x = -4$$

15)

$$x + k + \frac{9}{x} = 0$$

When the questions talks about "number of roots" or "number of solutions" we must use the discriminant. We are finding the actual solution, we are just talking about "how many" and this is our hint to use the solution.

Step 1: Make sure we have 0 on one side. We already have this, however we must multiply everything by x so we have a quadratic.

$$x^2 + kx + 9 = 0$$

Step 2: locate a, b,and c

$$1x^2 + kx + 9 = 0$$

a =number in front of $x^2 = 1$
b =number in front of $x = k$
c =number without an x or $x^2 = 9$

Note: Don't let the fact that we have a k confuse you. k is considered a constant here. x is the variable of the equation and that is the non-constant i.e. the variable that we don't include in a, b or c

$$x^2 + kx + 9 = 0$$

Step 3: We are given what case we are in, so we plug into the correct formula for that case

Case 1: $b^2 - 4ac > 0$ (2 real distinct/different roots)
Case 2: $b^2 - 4ac = 0$ (2 real repeated/double roots i.e. same/equal roots)
Case 3: $b^2 - 4ac < 0$ (no real roots)
Case 4: $b^2 - 4ac \geq 0$ (2 real roots i.e. doesn't mention whether distinct or repeated)

We are told 2 equal roots, hence we are in case 2:

$$k^2 - 4(1)(9) = 0$$

Step 4: Now we have an equation to solve to find p

$$k^2 - 4(1)(9) = 0$$
$$k^2 - 36 = 0$$
$$k^2 = 36$$
$$k = \pm 6$$

2.2.2 Inequalities

16)

$$x^2 - 3x + k^2 = 4$$

When the questions talks about "number of roots" or "number of solutions" we must use the discriminant. We are finding the actual solution, we are just talking about "how many" and this is our hint to use the solution.

Step 1: Make sure we have 0 on one side.

$$x^2 - 3x + k^2 - 4 = 0$$

Step 2: locate $a, b,$and c

$$1x^2 - 3x + k^2 - 4 = 0$$

a =number in front of $x^2 = 1$
b =number in front of $x = -3$
c =number without an x or $x^2 = k^2 - 4$

Note: Don't let the fact that we have a k confuse you. k is considered a constant here. x is the variable of the equation and that is the non-constant i.e. the variable that we don't include in a, b or c

$$x^2 - 3x + k^2 - 4 = 0$$

Step 3: We are given what case we are in, so we plug into the correct formula for that case

Case 1: $b^2 - 4ac > 0$ (2 real distinct/different roots)
Case 2: $b^2 - 4ac = 0$ (2 real repeated/double roots i.e. same/equal roots)
Case 3: $b^2 - 4ac < 0$ (no real roots)
Case 4: $b^2 - 4ac \geq 0$ (2 real roots i.e. doesn't mention whether distinct or repeated)

We are told 2 two distinct real solutions hence we are in case 1

$$(-3)^2 - 4(1)(k^2 - 4) > 0$$

Step 4: Now we have an inequality to solve to find k

$$(-3)^2 - 4(1)(k^2 - 4) > 0$$
$$9 - 4k^2 + 16 > 0$$
$$25 - 4k^2 > 0$$
$$4k^2 < 25$$
$$k^2 < \frac{25}{4}$$

Watch out for a really common mistake: Do not just guess the signs when solving a quadratic inequality, make sure to either graph or use a sign change test with a number line

$$-\frac{5}{2} < k < \frac{5}{2}$$

17)

$$x^2 + kx + 8 = k$$

When the questions talks about "number of roots" or "number of solutions" we must use the discriminant. We are finding the actual solution, we are just talking about "how many" and this is our hint to use the solution.

Step 1: Make sure we have 0 on one side.

$$x^2 + kx + 8 - k = 0$$

Step 2: locate $a, b,$and c

$$1x^2 + kx + 8 - k = 0$$

a =number in front of $x^2 = 1$
b =number in front of $x = k$
c =number without an x or $x^2 = 8 - k$

Note: Don't let the fact that we have a k confuse you. k is considered a constant here. x is the variable of the equation and that is the non-constant i.e. the variable that we don't include in a, b or c

$$x^2 + kx + 8 - k = 0$$

Step 3: We are given what case we are in, so we plug into the correct formula for that case

Case 1: $b^2 - 4ac > 0$ (2 real distinct/different roots)
Case 2: $b^2 - 4ac = 0$ (2 real repeated/double roots i.e. same/equal roots)
Case 3: $b^2 - 4ac < 0$ (no real roots)

Case 4: $b^2 - 4ac \geq 0$ (2 real roots i.e. doesn't mention whether distinct or repeated)

We are told 2 no real real solutions hence we are in case 3

$$k^2 - 4(1)(8 - k) < 0$$

Step 4: Now we have an inequality to solve to find k

$$k^2 - 4(1)(8 - k) < 0$$
$$k^2 - 32 + 4k < 0$$
$$k^2 + 4k - 32 < 0$$
$$(k - 4)(k + 8) < 0$$
$$k = 4, k = -8$$

Watch out for a really common mistake: Do not just guess the signs when solving a quadratic inequality, make sure to either graph or use a sign change test with a number line

$$-8 < k < 4$$

18)

$$x^2 - 2x + (3k^2 + 2k) = 0$$

When the questions talks about "number of roots" or "number of solutions" we must use the discriminant. We are finding the actual solution, we are just talking about "how many" and this is our hint to use the solution.

Step 1: Make sure we have 0 on one side. We already have this.

Step 2: locate $a, b,$and c

$$1x^2 - 2x + 3k^2 + 2k = 0$$

a =number in front of $x^2 = 1$
b =number in front of $x = -2$
c =number without an x or $x^2 = 3k^2 + 2k$

Note: Don't let the fact that we have a k confuse you. k is considered a constant here. x is the variable of the equation and that is the non-constant i.e. the variable that we don't include in a, b or c

$$x^2 - 2x + 3k^2 + 2k = 0$$

Step 3: We are given what case we are in, so we plug into the correct formula for that case

Case 1: $b^2 - 4ac > 0$ (2 real distinct/different roots)
Case 2: $b^2 - 4ac = 0$ (2 real repeated/double roots i.e. same/equal roots)
Case 3: $b^2 - 4ac < 0$ (no real roots)
Case 4: $b^2 - 4ac \geq 0$ (2 real roots i.e. doesn't mention whether distinct or repeated)

We are told 2 two distinct real solutions hence we are in case 1

$$(-2)^2 - 4(1)(3k^2 + 2k) > 0$$

Step 4: Now we have an inequality to solve to find k

$$(-2)^2 - 4(1)(3k^2 + 2k) > 0$$
$$4 - 12k^2 - 8k > 0$$
$$3k^2 + 2k - 1 < 0$$
$$(3k - 1)(k + 1) < 0$$
$$k = \frac{1}{3}, k = -1$$

Watch out for a really common mistake: Do not just guess the signs when solving a quadratic inequality, make sure to either graph or use a sign change test with a number line

$$-1 < k < \frac{1}{3}$$

2.3 Hidden Discriminant

2.3.1 Building your own equation

19)

When the questions talks about "number of roots" or "number of solutions" we must use the discriminant. "touch each other " can mean touch each other twice or once. Let's recall our cases

We need to represent the equation of intersecting\touching first before we can apply the discriminant. When the curves intersect we solve simultaneously so we can solve $y = k(2x^2 + 1)$ and $y = x^2 - 2x$ simultaneously

$$k(2x^2 + 1) = x^2 - 2x$$

Now we have an equation that we can use the discriminant on.

Step 1: Make sure we have 0 on one side. Here we have more than 1 x^2 term so we have to group it

$$2kx^2 + k = x^2 - 2x$$

$$(2k - 1)x^2 + 2x + k = 0$$

Step 2: locate a, b,and c

$$(2k - 1)x^2 + 2x + k = 0$$

a =number in front of $x^2 = 2k - 1$
b =number in front of $x = 2$
c =number without an x or $x^2 = k$

Note: Don't let the fact that we have a k confuse you. k is considered a constant here. x is the variable of the equation and that is the non-constant i.e. the variable that we don't include in a, b or c

$$(2k - 1)x^2 + 2x + k = 0$$

Step 3: Recall our cases

Case 1: $b^2 - 4ac > 0$ (2 real distinct/different roots)
Case 2: $b^2 - 4ac = 0$ (2 real repeated/double roots i.e. same/equal roots)
Case 3: $b^2 - 4ac < 0$ (no real roots)
Case 4: $b^2 - 4ac \geq 0$ (2 real roots i.e. doesn't mention whether distinct or

We are given touch each other which can mean touch each other twice or one. So we are in case 4 below

$$2^2 - 4(2k - 1)(k) \geq 0$$

Step 4: Now we have an inequality to solve to k

$$4 - 4\text{k}(2\text{k} - 1) \geq 0$$
$$4 - 8k^2 + 4k \geq 0$$
$$8k^2 - 4k - 4 \leq 0$$
$$2k^2 - k - 1 \leq 0$$
$$(2k + 1)(k - 1) \leq 0$$
$$k = -\frac{1}{2}, k = 1$$

Watch out for a really common mistake: Do not just guess the signs when solving a quadratic inequality, make sure to either graph or use a sign change test with a number line

$$-\frac{1}{2} \leq k \leq 1$$

20)

When the questions talks about "number of roots" or "number of solutions" we must use the discriminant. "does not cross or touch each other "means they do not intersect. Let's recall our cases

We need to represent the equation of intersecting\touching first before we can apply the discriminant. When the curves intersect we solve simultaneously so we can solve $y = 3x - 7$ and $y = 2px^2 - 6px + 4p$ simultaneously

$$3x - 7 = 2px^2 - 6px + 4p$$

Now we have an equation that we can use the discriminant on.

Step 1: Make sure we have 0 on one side. Here we have more than 1 x term so we have to group it

$$3x - 7 = 2px^2 - 6px + 4p$$
$$2px^2 - 6px - 3x + 4p + 7 = 0$$
$$2px^2 + (-6p - 3)x + (4p + 7) = 0$$

Step 2: locate a, b,and c

$$2px^2 + (-6p - 3)x + 4p + 7 = 0$$

a =number in front of $x^2 = 2p$
b =number in front of $x = -6p - 3$
c =number without an x or $x^2 = 4p + 7$

Note: Don't let the fact that we have a p confuse you. p is considered a constant here. x is the variable of the equation and that is the non-constant i.e. the variable that we don't include in a, b or c

$$(2p)x^2 + (-6p - 3)x + 4p + 7 = 0$$

Step 3: Recall our cases

Case 1: $b^2 - 4ac > 0$ (2 real distinct/different roots)
Case 2: $b^2 - 4ac = 0$ (2 real repeated/double roots i.e. same/equal roots)
Case 3: $b^2 - 4ac < 0$ (no real roots)
Case 4: $b^2 - 4ac \geq 0$ (2 real roots i.e. doesn't mention whether distinct or

We are given don't cross or touch each other. So we are in case 3:

$$(-6p - 3)^2 - 4(2p)(4p + 7) < 0$$

Step 4: Now we have an inequality to solve to p

$$(-6p - 3)^2 - 4(2p)(4p + 7) < 0$$
$$36p^2 + 9 + 36p - 32p^2 - 56p < 0$$
$$4p^2 - 20p + 9 < 0$$
$$(2p - 9)(2p - 1) < 0$$
$$k = \frac{9}{2}, k = \frac{1}{2}$$

Watch out for a really common mistake: Do not just guess the signs when solving a quadratic inequality, make sure to either graph or use a sign change test with a number line

$$\frac{1}{2} < p < \frac{9}{2}$$

21)

$$y = x(5 - x), 2y = 5x + 4$$

To show that the curve and the line do not intersect, we first have to find the discriminant and it should be less than 0.

We need to represent the equation of intersecting\touching first before we can apply the discriminant. When the curves intersect we solve simultaneously so we can solve $y = x(5 - x)$ and $2y = 5x + 4$ simultaneously

$$2(x(5 - x)) = 5x + 4$$
$$2(5x - x^2) = 5x + 4$$
$$10x - 2x^2 = 5x + 4$$

Now we have an equation that we can use the discriminant on.

Step 1: Make sure we have 0 on one side.

$$2x^2 - 5x + 4 = 0$$

Step 2: Locate $a, b,$and c

$$2x^2 - 5x + 4 = 0$$

a =number in front of $x^2 = 2$
b =number in front of x $= -5$
c =number without an x or $x^2 = 4$

Step 3: Plug into the formula $b^2 - 4ac$

$$(-5)^2 - 4(2)(4)$$

Step 4: Simplify

$$25 - 32 = -7$$

Step 5: Find the discriminant and decide which case you're in

Case 1: $b^2 - 4ac > 0$ (2 real distinct/different roots)
Case 2: $b^2 - 4ac = 0$ (2 real repeated/double roots i.e. same/equal roots)
Case 3: $b^2 - 4ac < 0$ (no real roots)
Case 4: $b^2 - 4ac \geq 0$ (2 real roots i.e. doesn't mention whether distinct or repeated)

$$-7 < 0$$

so we are in case 3 which means no real roots to the equation, hence the curve and the line do not intersect.

22)

We need to first represent the equation of intersecting\touching first before we can apply the discriminant. When the curves intersect we solve simultaneously so we can solve $kx + 2y - 3 = 0$ and $y = 3x^2 - 4x + 2$ simultaneously

$$kx + 2(3x^2 - 4x + 2) - 3 = 0$$

Now we have an equation that we can use the discriminant on.

Step 1: Make sure we have 0 on one side. We already have this however we should rearrange the equation to make it easier to locate $a, b,$and c in the next step. Here we have more than 1 x term so we have to group them

$$kx + 6x^2 - 8x + 4 - 3 = 0$$
$$6x^2 + kx - 8x + 1 = 0$$
$$6x^2 + (k - 8)x + 1 = 0$$

Step 2: locate $a, b,$and c

$$6x^2 + (k - 8)x + 1 = 0$$

a =number in front of $x^2 = 6$
b =number in front of $x = k - 8$

c =number without an x or x^2 $= 1$

Note: Don't let the fact that we have a k confuse you. k is considered a constant here. x is the variable of the equation and that is the non-constant i.e. the variable that we don't include in a, b or c

$$6x^2 + (k-8)x + 1 = 0$$

Step 3: Recall our cases

Case 1: $b^2 - 4ac > 0$ (2 real distinct/different roots)
Case 2: $b^2 - 4ac = 0$ (2 real repeated/double roots i.e. same/equal roots)
Case 3: $b^2 - 4ac < 0$ (no real roots)
Case 4: $b^2 - 4ac \geq 0$ (2 real roots i.e. doesn't mention whether distinct or

We are given touch each other only once. So we are in case 2

$$(k-8)^2 - 4(6)(1) = 0$$

Step 4: Now we have an equation to solve to k

$$k^2 - 16k + 64 - 24 = 0$$
$$k^2 - 16k + 40 = 0$$
$$k = \frac{-(-16) \pm \sqrt{(-16)^2 - 4(1)(40)}}{2(1)}$$
$$k = 8 + 2\sqrt{6}, k = 8 - 2\sqrt{6}$$

The question asks for exact values so leave them in surd form.

23)

We need to first represent the equation of intersecting\touching first before we can apply the discriminant. "touch each other " can mean touch each other twice or once. When the curves intersect we solve simultaneously so we can solve $y = x^2 - kx$ and $y = 3(k+1) + kx - x^2$ simultaneously

$$x^2 - kx = 3(k+1) + kx - x^2$$

Now we have an equation that we can use the discriminant on. Before we do this, we will first determine the value of k for which the curves cross the y axis.

Step 1: Make sure we have 0 on one side. We already have this however we should rearrange the equation to make it easier to locate a, b,and c in the next step.

$$x^2 - kx = 3k + 3 + kx - x^2$$
$$2x^2 - 2kx - 3k - 3 = 0$$

Step 2: locate a, b,and c

$$2x^2 - 2kx - 3k - 3 = 0$$

a =number in front of x^2 $= 2$
b =number in front of x $= -2k$
c =number without an x or x^2 $= -3k - 3$

Note: Don't let the fact that we have a k confuse you. k is considered a constant here. x is the variable of the equation and that is the non-constant i.e. the variable that we don't include in a, b or c

$$2x^2 - 2kx - 3k - 3 = 0$$

Step 3: Recall our cases

Case 1: $b^2 - 4ac > 0$ (2 real distinct/different roots)
Case 2: $b^2 - 4ac = 0$ (2 real repeated/double roots i.e. same/equal roots)
Case 3: $b^2 - 4ac < 0$ (no real roots)
Case 4: $b^2 - 4ac \geq 0$ (2 real roots i.e. doesn't mention whether distinct or

We have found 1 intersection already so there's one left. So we are in case 2 below

$$(-2k)^2 - 4(2)(-3k - 3) = 0$$

Step 4: Now we have an equation to solve in k

$$4k^2 + 24k + 24 = 0$$
$$k^2 + 6k + 6 = 0$$

$$k = \frac{-(6) \pm \sqrt{(6)^2 - 4(1)(6)}}{2(1)}$$
$$k = -3 \pm \sqrt{3}$$

i.

When they cross on the y axis, $x = 0$.

$$x^2 - kx = 3k + 3 + kx - x^2$$

$$2x^2 - 2kx - 3k - 3 = 0$$
$$2(0)^2 - 2k(0) - 3k - 3 = 0$$
$$-3k - 3 = 0$$
$$k = -1$$

24)

When the questions talks about "number of roots" or "number of solutions" we must use the discriminant. "does not cross or touch each other "means they do not intersect. Let's recall our cases

We need to represent the equation of intersecting\touching first before we can apply the discriminant. When the curves intersect we solve simultaneously so we can solve $f(x) = m - \frac{1}{x}$ and , $y = x - m$ simultaneously

$$m - \frac{1}{x} = x - m$$

Now we have an equation that we can use the discriminant on.

Step 1: Make sure we have 0 on one side.

$$x^2 - mx = mx - 1$$
$$x^2 - 2mx + 1 = 0$$

Step 2: locate a, b,and c

$$1x^2 - 2mx + 1 = 0$$

a =number in front of $x^2 = 1$
b =number in front of $x = -2m$
c =number without an x or $x^2 = 1$

Note: Don't let the fact that we have a p confuse you. p is considered a constant here. x is the variable of the equation and that is the non-constant i.e. the variable that we don't include in a, b or c

$$x^2 - 2mx + 1 = 0$$

Step 3: Recall our cases

Case 1: $b^2 - 4ac > 0$ (2 real distinct/different roots)
Case 2: $b^2 - 4ac = 0$ (2 real repeated/double roots i.e. same/equal roots)
Case 3: $b^2 - 4ac < 0$ (no real roots)
Case 4: $b^2 - 4ac \geq 0$ (2 real roots i.e. doesn't mention whether distinct or

We are given that they cross in two distinct points so. So we are in case 1:

$$(-2m)^2 - 4(1)(1) > 0$$

Step 4: Now we have an inequality to solve in m

$$4m^2 - 4 > 0$$
$$m^2 - 1 > 0$$
$$m^2 > 1$$
$$m = \pm 1$$

Watch out for a really common mistake: Do not just guess the signs when solving a quadratic inequality, make sure to either graph or use a sign change test with a number line

$$m < -1, m > 1$$

25)

When the questions talks about "number of roots" or "number of solutions" we must use the discriminant. "does not cross or touch each other "means they do not intersect. Let's recall our cases

We need to represent the equation of intersecting\touching first before we can apply the discriminant. When the curves intersect we solve simultaneously so we can solve $y = -3x + c$ and $y = \frac{1}{x} + 5$ simultaneously

$$-3x + c = \frac{1}{x} + 5$$

Now we have an equation that we can use the discriminant on.

Step 1: Make sure we have 0 on one side. Here we have more than 1 x term so we have to group them

$$-3x + c = \frac{1}{x} + 5$$
$$-3x^2 + cx = 1 + 5x$$
$$3x^2 + 5x - cx + 1 = 0$$
$$3x^2 + (5 - c)x + 1 = 0$$

Step 2: locate $a, b,$and c

$$3x^2 + (5 - c)x + 1 = 0$$

a =number in front of $x^2 = 3$
b =number in front of $x = 5 - c$
c =number without an x or $x^2 = 1$

Note: Don't let the fact that we have a c confuse you. c is considered a constant here. x is the variable of the equation and that is the non-constant i.e. the variable that we don't include in a, b or c

$$3x^2 + (5 - c)x + 1 = 0$$

Step 3: Recall our cases

Case 1: $b^2 - 4ac > 0$ (2 real distinct/different roots)
Case 2: $b^2 - 4ac = 0$ (2 real repeated/double roots i.e. same/equal roots)
Case 3: $b^2 - 4ac < 0$ (no real roots)
Case 4: $b^2 - 4ac \geq 0$ (2 real roots i.e. doesn't mention whether distinct or

We are given that they meet at two distinct points so. So we are in case 1:

$$(5 - c)^2 - 4(3)(1) > 0$$

Step 4: Now we have an inequality to solve to c

$$(5 - c)^2 - 4(3)(1) > 0$$
$$(5 - c)^2 - 12 > 0$$
$$(5 - c)^2 > 12$$
$$5 - c = \pm\sqrt{12}$$

Watch out for a really common mistake: Do not just guess the signs when solving a quadratic inequality, make sure to either graph or use a sign change test with a number line

$$5 - c < -\sqrt{12} \qquad 5 - c > \sqrt{12}$$
$$c > 5 + 2\sqrt{3} \qquad c < 5 - 2\sqrt{3}$$

26)

i.

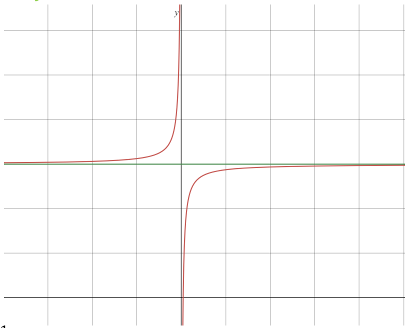
$$y = k - \frac{1}{2x}$$

This curve is a transformation of the curve $y = \frac{1}{x}$

Reflected over the x-axis

Enlarged by a scale factor of $\frac{1}{2}$

Translated by k in the positive y direction



Equation of asymptote of $y = \frac{1}{x}$ is $y = 0$

Reflected over x-axis:

$$y = 0$$

Enlarged by scale factor of $\frac{1}{2}$:

$$y = 0$$

Finally translate by k in the positive y direction:

$$y = k$$

ii.

$$y = x(5 - x),\ 2y = 5x + 4$$

When the question mentions the graphs intersect or in this case a straight line cutting into a curve, we must use the discriminant.

We need to represent the equation of intersecting\touching first before we can apply the discriminant. When the curves intersect we solve simultaneously so we can solve $y = 2x + 3$ and $y = k - \frac{1}{2x}$ simultaneously

$$2x + 3 = k - \frac{1}{2x}$$
$$2x(2x + 3) = 2kx - 1$$
$$4x^2 + 6x = 2kx - 1$$

Now we have an equation that we can use the discriminant on.

Step 1: Make sure we have 0 on one side.

$$4x^2 + 6x - 2kx + 1 = 0$$
$$4x^2 + (6 - 2k)x + 1 = 0$$

Step 2: Locate a, b,and c

$$4x^2 + (6 - 2k)x + 1 = 0$$

a =number in front of $x^2 = 4$

b =number in front of $x = 6 - 2k$

c =number without an x or $x^2 = 1$

Note: Don't let the fact that we have a k confuse you. k is considered a constant here. x is the variable of the equation and that is the non-constant i.e. the variable that we don't include in a, b or c

$$4x^2 + (6 - 2k)x + 1 = 0$$

Step 3: Recall our cases

Case 1: $b^2 - 4ac > 0$ (2 real distinct/different roots)

Case 2: $b^2 - 4ac = 0$ (2 real repeated/double roots i.e. same/equal roots)

Case 3: $b^2 - 4ac < 0$ (no real roots)

Case 4: $b^2 - 4ac \geq 0$ (2 real roots i.e. doesn't mention whether distinct or

We are given that they meet at two distinct points. So we are in case 1:

$$(6-2k)^2 - 4(4)(1) > 0$$

Step 4: Now we have an inequality to solve to c

$$(6-2k)^2 - 4(4)(1) > 0$$
$$36 - 24k + 4k^2 - 16 > 0$$
$$4k^2 - 24k + 20 > 0$$
$$k^2 - 6k + 5 > 0$$
$$(k-5)(k-1) > 0$$
$$k = 5, k = 1$$

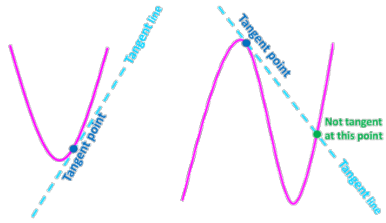
Watch out for a really common mistake: Do not just guess the signs when solving a quadratic inequality, make sure to either graph or use a sign change test with a number line

$$\boldsymbol{k < 1, k > 5}$$
$$\{\boldsymbol{k: k < 1}\} \cup \{\boldsymbol{k: k > 5}\}$$

2.3.2 Tangent to curve

27)

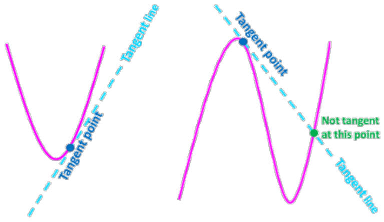
There are 2 ways to do this

Way 1: Use discriminant	**Way 2: Use tangent definition that gradients are the same and curves intersect**
Note: This method only works if we have a quadratic, since a tangent can cross a curve more than once if not a quadratic (also the discriminant can only be used on a quadratic)  Tangent means meets one hence $b^2 - 4ac = 0$ Let's first represent the curves intersecting/meeting $y = mx + 5, y = 4 - x^2$ $mx + 5 = 4 - x^2$ $x^2 + mx + 1 = 0$ Let's find the discriminant $b^2 - 4ac = m^2 - 4(1)(1) = m^2 - 4$ Meet once hence $b^2 - 4ac = 0$ $m^2 - 4 = 0$ $m^2 = 4$ $m = \pm 2$	$y = mx + 5, y = 4 - x^2$ Tangent means same gradient: Differentiating the line: $\frac{dy}{dx} = m$ Differentiating the curve we get: $\frac{dy}{dx} = -2x$ Tangent means gradients are equal: $m = -2x$ ① Tangent means the equations intersect $y = mx + 5, y = 4 - x^2$ $mx + 5 = 4 - x^2$ $x^2 + mx + 1 = 0$ ② Solve ① and ② simultaneously $x^2 + (-2x)x + 1 = 0$ $x^2 = 1$ $x = \pm 1$ $m = \pm 2$

28)

There are 2 ways to do this

Way 1: Use discriminant	**Way 2: Use tangent definition that gradients are the same and curves intersect**
Note: This method only works if we have a quadratic, since a tangent can cross a curve more than once if not a quadratic (also the discriminant can only be used on a quadratic)	$y = 2x - 3, y = x^2 - 2$ Tangent means same gradient: Differentiating the line: $\frac{dy}{dx} = 2$ Differentiating the curve we get: $\frac{dy}{dx} = 2x$ Tangent means gradients are equal: $2x = 2$ $x = 1$ ①



Tangent means meets one hence $b^2 - 4ac = 0$

Let's first represent the curves intersecting/meeting

$y = 2x - 3,\ y = x^2 - 2$

$2x - 3 = x^2 - 2$

$x^2 - 2x + 1 = 0$

Let's find the discriminant

$b^2 - 4ac = (-2)^2 - 4(1)(1) = 4 - 4 = 0$

$b^2 - 4ac = 0$, hence the line is a tangent

Tangent means the equations intersect

$y = 2x - 3,\ y = x^2 - 2$

$2x - 3 = x^2 - 2$

$x^2 - 2x + 1 = 0$ ②

Solve ① and ② simultaneously

$(1)^2 - 2(1) + 1 = 0$

$1 - 2 + 1 = 0$

$0 = 0$

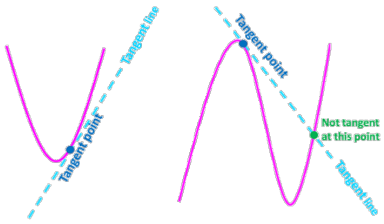
hence the line is a tangent

29)

There are 2 ways to do this

Way 1: Use discriminant

Note: This method only works if we have a quadratic, since a tangent can cross a curve more than once if not a quadratic (also the discriminant can only be used on a quadratic)



Tangent means meets one hence $b^2 - 4ac = 0$

Let's first represent the curves intersecting/meeting

$y = 4x - 9,\ y = 4x(x - 2)$

$4x - 9 = 4x(x - 2)$

$4x - 9 = 4x^2 - 8x$

$4x^2 - 12x + 9 = 0$

Let's find the discriminant

$b^2 - 4ac = (-12)^2 - 4(4)(9) = 144 - 144 = 0$

$b^2 - 4ac = 0$, hence the line is a tangent

Way 2: Use tangent definition that gradients are the same and curves intersect

$y = 4x - 9,\ y = 4x(x - 2)$

Tangent means same gradient:

Differentiating the line: $\frac{dy}{dx} = 4$

Differentiating the curve we get:

$y = 4x^2 - 8x$

$\frac{dy}{dx} = 8x - 8$

Tangent means gradients are equal: $8x - 8 = 4$

$x = \frac{3}{2}$ ①

Tangent means the equations intersect:

$y = 4x - 9,\ y = 4x(x - 2)$

$4x - 9 = 4x(x - 2)$

$4x - 9 = 4x^2 - 8x$

$4x^2 - 12x + 9 = 0$ ②

Solve ① and ② simultaneously

$4\left(\frac{3}{2}\right)^2 - 12\left(\frac{3}{2}\right) + 9 = 0$

$9 - 18 + 9 = 0$

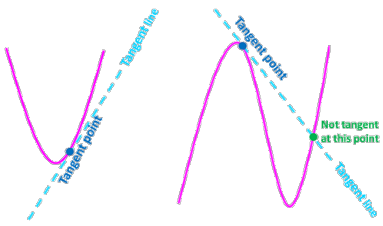
$0 = 0$

hence the line is a tangent

30)

Way 1: Use discriminant

Note: This method only works if we have a quadratic, since a tangent can cross a curve more than once if not a quadratic (also the discriminant can only be used on a quadratic)



Tangent means meets one hence $b^2 - 4ac = 0$

Let's first represent the curves intersecting/meeting

$y = 2x + k,\ y = x^2 - 4x + 2$

$2x + k = x^2 - 4x + 2$

$x^2 - 6x + 2 - k = 0$

Way 2: Use tangent definition that gradients are the same and curves intersect

$y = 2x + k,\ y = x^2 - 4x + 2$

Tangent means same gradient:

Differentiating the line: $\frac{dy}{dx} = 2$

Differentiating the curve we get: $\frac{dy}{dx} = 2x - 4$

Tangent means gradients are equal: $2x - 4 = 2$

$x = 3$ ①

Tangent means the equations intersect:

$y = 2x + k,\ y = x^2 - 4x + 2$

$2x + k = x^2 - 4x + 2$

$x^2 - 6x + 2 - k = 0$ ②

Solve ① and ② simultaneously

$(3)^2 - 6(3) + 2 - k = 0$

$9 - 18 + 2 - k = 0$

$k = -7$

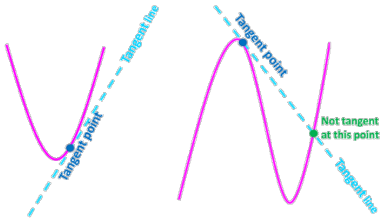
Let's find the discriminant. As it's a tangent,

$$b^2 - 4ac = (-6)^2 - 4(1)(2-k)$$
$$36 - 8 + 4k = 0$$
$$4k = -28$$
$$k = -7$$

31)

Way 1: Use discriminant

Note: This method only works if we have a quadratic, since a tangent can cross a curve more than once if not a quadratic (also the discriminant can only be used on a quadratic)



Tangent means meets one hence $b^2 - 4ac = 0$

Let's first represent the curves intersecting/meeting

$$y = 4x + c,\ y = 2x^2 + 8x + 3$$
$$4x + c = 2x^2 + 8x + 3$$
$$2x^2 + 4x + 3 - c = 0$$

Let's find the discriminant. As it's a tangent, $b^2 - 4ac = 0$

$$b^2 - 4ac = (4)^2 - 4(2)(3-c)$$
$$16 - 24 + 8c = 0$$
$$8c = 8$$
$$c = 1$$

Way 2: Use tangent definition that gradients are the same and curves intersect

$$y = 4x + c,\ y = 2x^2 + 8x + 3$$

Tangent means same gradient:

Differentiating the line: $\frac{dy}{dx} = 4$

Differentiating the curve we get: $\frac{dy}{dx} = 4x + 8$

Tangent means gradients are equal: $4x + 8 = 4$

$x = -1$ (1)

Tangent means the equations intersect:

$$y = 4x + c,\ y = 2x^2 + 8x + 3$$
$$4x + c = 2x^2 + 8x + 3$$
$$2x^2 + 4x + 3 - c = 0 \quad (2)$$

Solve (1) and (2) simultaneously

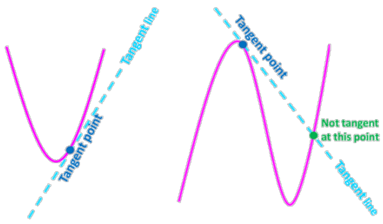
$$2(-1)^2 + 4(-1) + 3 - c = 0$$
$$2 - 4 + 3 - c = 0$$

$c = 1$

32)

Way 1: Use discriminant

Note: This method only works if we have a quadratic, since a tangent can cross a curve more than once if not a quadratic (also the discriminant can only be used on a quadratic)



Tangent means meets one hence $b^2 - 4ac = 0$

Let's first represent the curves intersecting/meeting

$$y = 3x + k,\ y = \frac{1}{3}x^2 + 8$$
$$3x + k = \frac{1}{3}x^2 + 8$$
$$9x + 3k = x^2 + 24$$
$$x^2 - 9x + 24 - 3k = 0$$

Let's find the discriminant. As it's a tangent, $b^2 - 4ac = 0$

$$b^2 - 4ac = (-9)^2 - 4(1)(24 - 3k)$$
$$81 - 96 + 12k = 0$$
$$12k = 15$$
$$k = \frac{5}{4}$$

Way 2: Use tangent definition that gradients are the same and curves intersect

$$y = 3x + k,\ y = \frac{1}{3}x^2 + 8$$

Tangent means same gradient:

Differentiating the line: $\frac{dy}{dx} = 3$

Differentiating the curve we get: $\frac{dy}{dx} = \frac{2}{3}x$

Tangent means gradients are equal: $\frac{2}{3}x = 3$

$x = \frac{9}{2}$ (1)

Tangent means the equations intersect:

$$y = 3x + k,\ y = \frac{1}{3}x^2 + 8$$
$$3x + k = \frac{1}{3}x^2 + 8$$
$$9x + 3k = x^2 + 24$$
$$x^2 - 9x + 24 - 3k = 0 \quad (2)$$

Solve (1) and (2) simultaneously

$$\left(\frac{9}{2}\right)^2 - 9\left(\frac{9}{2}\right) + 24 - 3k = 0$$
$$\frac{81}{4} - \frac{81}{2} + 24 - 3k = 0$$
$$3k = \frac{15}{4}$$
$$k = \frac{5}{4}$$

33)

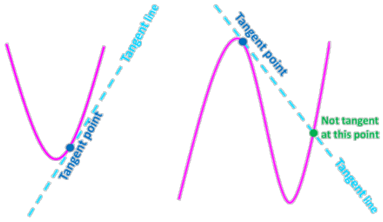
Way 1: Use discriminant

Note: This method only works if we have a quadratic, since a tangent can cross a curve more than once if not a quadratic (also the discriminant can only be used on a quadratic)

Way 2: Use tangent definition that gradients are the same and curves intersect

$$y = mx + k,\ y = 4 - x^2$$

Tangent means same gradient:



Tangent means meets one hence $b^2 - 4ac = 0$

Let's first represent the curves intersecting/meeting

$$y = mx + k,\ y = 4 - x^2$$
$$mx + k = 4 - x^2$$
$$x^2 + mx + 1 = 0$$

Let's find the discriminant. As it's a tangent, $b^2 - 4ac = 0$

$$b^2 - 4ac = m^2 - 4(1)(1)$$
$$m^2 = 4$$
$$m = \pm 2$$

Differentiating the line: $\frac{dy}{dx} = m$

Differentiating the curve we get: $\frac{dy}{dx} = -2x$

Tangent means gradients are equal: $-2x = m$

$$m = -2x \ (1)$$

Tangent means the equations intersect:

$$y = mx + k,\ y = 4 - x^2$$
$$mx + k = 4 - x^2$$
$$x^2 + mx + 1 = 0 \ (2)$$

Solve (1) and (2) simultaneously

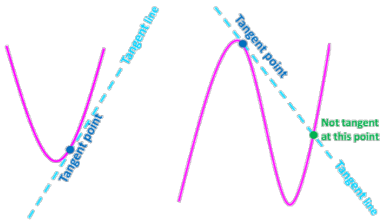
$$x^2 + (-2x)x + 1 = 0$$
$$-x^2 = -1$$
$$x = \pm 1$$
$$m = -2x = \pm 2$$

34)

i.

Way 1: Use discriminant

Note: This method only works if we have a quadratic, since a tangent can cross a curve more than once if not a quadratic (also the discriminant can only be used on a quadratic)



Tangent means meets one hence $b^2 - 4ac = 0$

Let's first represent the curves intersecting/meeting

$$y = 5x - 3,\ y = kx^2 - 3x + 5$$
$$5x - 3 = kx^2 - 3x + 5$$
$$kx^2 - 8x + 8 = 0$$

Let's find the discriminant. As it's a tangent, $b^2 - 4ac = 0$

$$b^2 - 4ac = (-8)^2 - 4(k)(8)$$
$$64 - 32k = 0$$
$$k = 2$$

Way 2: Use tangent definition that gradients are the same and curves intersect

$$y = 5x - 3,\ y = kx^2 - 3x + 5$$

Tangent means same gradient:

Differentiating the line: $\frac{dy}{dx} = 5$

Differentiating the curve we get: $\frac{dy}{dx} = 2kx - 3$

Tangent means gradients are equal: $2kx - 3 = 5$

$$x = \frac{4}{k} \ (1)$$

Tangent means the equations intersect:

$$y = 5x - 3,\ y = kx^2 - 3x + 5$$
$$5x - 3 = kx^2 - 3x + 5$$
$$kx^2 - 8x + 8 = 0 \ (2)$$

Solve (1) and (2) simultaneously

$$k\left(\frac{k}{4}\right)^2 - 8\left(\frac{k}{4}\right) + 8 = 0$$
$$16k - \frac{32}{k} + 8 = 0$$
$$16k^2 + 8k - 32 = 0$$
$$8k^2 + 4k - 16 = 0$$
$$(k - 2)(k + 8) = 0$$
$$k = 2, -8$$
$$k = 2$$

ii.

When $k = 2$,

Substitute $k = 2$ to $y = kx^2 - 3x + 5$

$$y = 2x^2 - 3x + 5$$
$$\frac{dy}{dx} = 4x - 3$$

The gradient of the curve at point A should be 5 since $y = 5x - 3$ is the equation of tangent at point A.

$$\frac{dy}{dx} = 5$$
$$4x - 3 = 5$$
$$4x = 8$$
$$x = 2$$

When $x = 2$,

Substitute $x = 2$ to $y = 5x - 3$

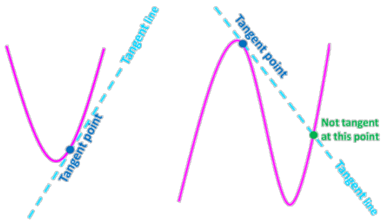
$$y = 5(2) - 3 = 10 - 3 = 7$$

$\therefore$ A $= (2, 7)$

35)

Way 1: Use discriminant

Note: This method only works if we have a quadratic, since a tangent can cross a curve more than once if not a quadratic (also the discriminant can only be used on a quadratic)



Tangent means meets one hence $b^2 - 4ac = 0$

Let's first represent the curves intersecting/meeting

$$y = c - 3x,\ y = \frac{3}{x}$$

$$c - 3x = \frac{3}{x}$$

$$3x^2 - cx + 3 = 0$$

Let's find the discriminant. As it's a tangent, $b^2 - 4ac = 0$

$$b^2 - 4ac = (-c)^2 - 4(3)(3)$$

$$c^2 - 36 = 0$$

$$c^2 = 36$$

$$c = \pm 6$$

Way 2: Use tangent definition that gradients are the same and curves intersect

$$y = c - 3x,\ y = \frac{3}{x}$$

Tangent means same gradient:

Differentiating the line: $\frac{dy}{dx} = -3$

Differentiating the curve we get: $\frac{dy}{dx} = -\frac{3}{x^2}$

Tangent means gradients are equal: $-\frac{3}{x^2} = -3$

$$x = \pm 1 \ ①$$

Tangent means the equations intersect:

$$y = c - 3x,\ y = \frac{3}{x}$$

$$c - 3x = \frac{3}{x}$$

$$3x^2 - cx + 3 = 0 \ ②$$

Solve ① and ② simultaneously

$$3(\pm 1)^2 - c(\pm 1) + 3 = 0$$

$$3 \pm c + 3 = 0$$

$$6 \pm c = 0$$

$$\pm c = 6$$

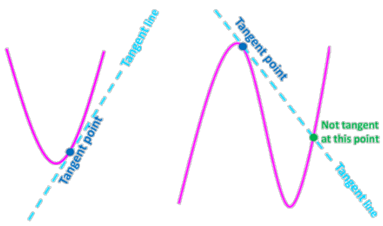
$$c = \pm 6$$

36)

$$y = 3x + k,\ x^2 + xy + 16 = 0$$

i.

Use discriminant



Tangent means meets one hence $b^2 - 4ac = 0$

Let's first represent the curves intersecting/meeting

$$y = 3x + k,\ x^2 + xy + 16 = 0$$

$$x^2 + x(3x + k) + 16 = 0$$

$$x^2 + 3x^2 + kx + 16 = 0$$

$$4x^2 + kx + 16 = 0$$

Let's find the discriminant. As it's a tangent, $b^2 - 4ac = 0$

$$b^2 - 4ac = k^2 - 4(4)(16) = 0$$

$$k^2 - 256 = 0$$

$$k^2 = 256$$

$$k = \pm 16$$

ii.

When $k = 16$,

$$4x^2 + 16x + 16 = 0$$

$$x^2 + 4x + 4 = 0$$

$$(x + 2)^2 = 0$$

$$x = -2$$

Substitute $x = -2$ into $y = 3x + 16$,

$$y = 3(-2) + 16 = -6 + 16 = 10$$

∴ The coordinates of the point of contact when $k = 16$ are $(\mathbf{-2}, \mathbf{10})$.
When $k = -16$,

$$4x^2 - 16x + 16 = 0$$
$$x^2 - 4x + 4 = 0$$
$$(x-2)^2 = 0$$
$$x = 2$$

Substitute $x = 2$ into $y = 3x - 16$,

$$y = 3(2) - 16 = 6 - 16 = -10$$

∴ The coordinates of the point of contact when $k = -16$ are $(\mathbf{2}, \mathbf{-10})$.

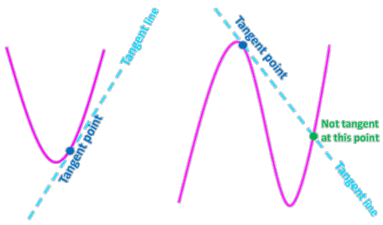
37)

$$y = x^2 + kx + k, y = -x^2 + 2x - 4$$

i.

Way 1: Use discriminant since we have a quadratic

Note: This method only works if we have a quadratic, since a tangent can cross a curve more than once if not a quadratic (also the discriminant can only be used on a quadratic)



Tangent means meets one hence $b^2 - 4ac = 0$
Let's first represent the curves intersecting/meeting

$$y = x^2 + kx + k, y = -x^2 + 2x - 4$$
$$x^2 + kx + k = -x^2 + 2x - 4$$
$$2x^2 + kx - 2x + k + 4 = 0$$
$$2x^2 + (k-2)x + (k+4) = 0$$

Let's find the discriminant. They meet at P, $b^2 - 4ac = 0$

$$b^2 - 4ac = (k-2)^2 - 4(2)(k+4)$$
$$k^2 - 4k + 4 - 8k - 32 = 0$$
$$k^2 - 12k - 28 = 0$$
$$(k-14)(k+2) = 0$$
$$k = 14, k = -2$$

Since $k < 0$, $k = -2$.

ii.

When $k = -2$,

$$2x^2 + (-2-2)x + 2 = 0$$
$$2x^2 - 4x + 2 = 0$$
$$x^2 - 2x + 1 = 0$$
$$(x-1)^2 = 0$$
$$x = 1$$

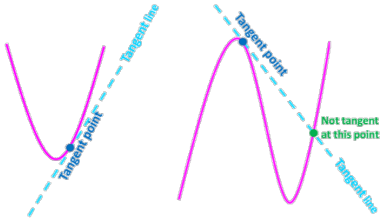
Substitute $x = 1$ into $y = x^2 - 2x - 2$,

$$y = 1^2 - 2(1) - 2 = 1 - 2 - 2 = -3$$

∴ P $= (1, -3)$

38)

Way 1: Use discriminant since we have a quadratic



Tangent means meets one hence $b^2 - 4ac = 0$

Let's first represent the curves intersecting/meeting

$$y = m(x - m), (1 - x)y = 1$$
$$(1 - x)(m)(x - m) = 1$$
$$(1 - x)(mx - m^2) = 1$$
$$mx - m^2 - mx^2 + m^2x = 1$$
$$mx^2 - mx - m^2x + m^2 + 1 = 0$$
$$mx^2 + (-m - m^2)x + (m^2 + 1) = 0$$

Let's find the discriminant. As it's a tangent, $b^2 - 4ac = 0$

$$b^2 - 4ac = (-m - m^2)^2 - 4(m)(m^2 + 1) = 0$$
$$m^2 + m^4 + 2m^3 - 4m^3 - 4m = 0$$
$$m^4 - 2m^3 + m^2 - 4m = 0$$
$$m^3 - 2m^2 + m - 4 = 0$$
$$m = 2.31$$

Substitute $m = 2.31$ into $mx^2 + (-m - m^2)x + (m^2 + 1) = 0$

$$x = 1.66$$

Substitute $m = 2.31$ and $x = 1.66$ into $y = m(x - m)$ or $(1 - x)y = 1$

$$y = -1.52$$

$$(1.66, -1.52)$$

Way 2: Use tangent definition that gradients are the same and curves intersect

$$y = m(x - m), (1 - x)y = 1$$

Tangent means same gradient:

Differentiating the line:

$$y = m(x - m)$$
$$y = mx - m^2$$
$$\frac{dy}{dx} = m$$

Differentiating the curve:

$$(1 - x)y = 1$$
$$y = \frac{1}{1 - x} = (1 - x)^{-1}$$
$$\frac{dy}{dx} = -(1 - x)^{-2}(-1) = \frac{1}{(1 - x)^2}$$

Tangent means gradients are equal:

$$\frac{1}{(1 - x)^2} = \text{m}$$
$$m = \frac{1}{(1-x)^2} \quad (1)$$

Tangent means the equations intersect

$$y = m(x - m), (1 - x)y = 1$$
$$(1 - x)(m)(x - m) = 1$$
$$\frac{1}{1-x} = m(x - m) \quad (2)$$

Solve (1) and (2) simultaneously

$$\frac{1}{1 - x} = \frac{1}{(1 - x)^2}\left(x - \frac{1}{(1 - x)^2}\right)$$
$$(1 - x)^3 - x(1 - x)^2 + 1 = 0$$
$$x \neq 1$$
$$x = 1.66$$
$$y = \frac{1}{1 - x} = -1.52$$

$$(1.66, -1.52)$$

$$m = \frac{1}{(1 - x)^2} = \frac{1}{(1 - 1.66)^2} = 2.31$$

39)

$$y = x^2 - 4x + 4, y = mx$$

i.

When $m = 1$ the curves intersect,

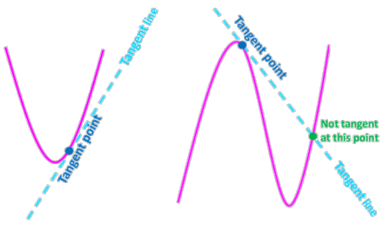
$$x^2 - 4x + 4 = (1)x$$
$$x^2 - 4x + 4 = x$$
$$x^2 - 5x + 4 = 0$$
$$(x - 4)(x - 1) = 0$$
$$x = 4, x = 1$$
$$y = 4, y = 1$$
$$\text{A} = (4, 4), \text{B} = (1, 1)$$

Now we have the two coordinates, we must find the midpoint

$$\text{Midpoint AB} = \left(\frac{4+1}{2}, \frac{4+1}{2}\right) = \left(\frac{5}{2}, \frac{5}{2}\right)$$

ii.

Use discriminant since we have a quadratic



Tangent means meets one hence $b^2-4ac=0$
Let's first represent the curves intersecting/meeting

$$y=x^2-4x+4,\ y=mx$$
$$x^2-4x+4=mx$$
$$x^2-4x-mx+4=0$$
$$x^2+(-4-m)x+4=0$$

Let's find the discriminant. As it's a tangent, $b^2-4ac=0$

$$b^2-4ac=(-4-m)^2-4(m)(m^2+1)=0$$
$$16+m^2+8m-16=0$$
$$m^2+8m=0$$
$$m(m+8)=0$$
$$m=0, m=-8$$

Since $m\neq 0$, $m=-8$.

Substitute $m=-8$ into $x^2+(-4-m)x+4=0$

$$x^2+(-4-(-8))x+4=0$$
$$x^2+4x+4=0$$
$$(x+2)^2=0$$
$$x=-2$$

Substitute $m=-8$ and $x=-2$ into $y=mx$

$$y=(-8)(-2)$$
$$y=16$$

$$(-2,16)$$

3 Gold

3.1 Finding the number of roots

40)

$$x^2 - kx - 5 = 0$$

When the questions talks about "number of roots" or "number of solutions" we must use the discriminant. We aren't finding the actual solution, we are just talking about "how many" and this is our hint to use the solution.

Step 1: Make sure we have 0 on one side. We already have this

Step 2: Locate a, b,and c

$$1x^2 - kx - 5 = 0$$

a =number in front of $x^2 = 1$
b =number in front of $x = -k$
c =number without an x or $x^2 = -5$

Step 3: Plug into the formula $b^2 - 4ac$

$$(-k)^2 - 4(1)(-5)$$

Step 4: Simplify

$$k^2 + 20$$

Step 5: Find the discriminant and decide which case you're in

Case 1: $b^2 - 4ac > 0$ (2 real distinct/different roots)
Case 2: $b^2 - 4ac = 0$ (2 real repeated/double roots i.e. same/equal roots)
Case 3: $b^2 - 4ac < 0$ (no real roots)
We can combine case 1 and 3 if the question just says real and doesn't mention whether distinct of the repeated
Case 4: $b^2 - 4ac \geq 0$ (2 real roots i.e. doesn't mention whether distinct or repeated)

As k^2 is always positive, $k^2 + 20 > 0$ so we are in case 1 which means 2 real distinct roots

41)

$$px^2 + qx - 4p = 0$$

When the questions talks about "number of roots" or "number of solutions" we must use the discriminant. We aren't finding the actual solution, we are just talking about "how many" and this is our hint to use the solution.

Step 1: Make sure we have 0 on one side. We already have this

Step 2: Locate a, b,and c

$$px^2 + qx - 4p = 0$$

a =number in front of $x^2 = p$
b =number in front of $x = q$
c =number without an x or $x^2 = -4p$

Step 3: Plug into the formula $b^2 - 4ac$

$$q^2 - 4(p)(-4p)$$

Step 4: Simplify

$$q^2 + 16p^2$$

Step 5: Find the discriminant and decide which case you're in

Case 1: $b^2 - 4ac > 0$ (2 real distinct/different roots)
Case 2: $b^2 - 4ac = 0$ (2 real repeated/double roots i.e. same/equal roots)
Case 3: $b^2 - 4ac < 0$ (no real roots)
We can combine case 1 and 3 if the question just says real and doesn't mention whether distinct of the repeated
Case 4: $b^2 - 4ac \geq 0$ (2 real roots i.e. doesn't mention whether distinct or repeated)

As q^2 and p^2 are always positive, $q^2 + 16p^2 > 0$ so we are in case 1 which means 2 real distinct roots

3.2 Given number of roots, solve for an unknown

3.2.1 Equalities

42)

$$px^2 + (10 - p)x + \frac{5}{4}p - 5 = 0$$

When the questions talks about "number of roots" or "number of solutions" we must use the discriminant. We are finding the actual solution, we are just talking about "how many" and this is our hint to use the solution.

Step 1: Make sure we have 0 on one side. We already have this.

Step 2: locate a, b,and c

$$px^2 + (10 - p)x + \frac{5}{4}p - 5 = 0$$

a =number in front of $x^2 = p$
b =number in front of $x = 10 - p$
c =number without an x or $x^2 = \frac{5}{4}p - 5$

Note: Don't let the fact that we have a p confuse you. p is considered a constant here. x is the variable of the equation and that is the non-constant i.e. the variable that we don't include in a, b or c

$$px^2 + (10 - p)x + \frac{5}{4}p - 5 = 0$$

Step 3: We are given what case we are in, so we plug into the correct formula for that case

Case 1: $b^2 - 4ac > 0$ (2 real distinct/different roots)
Case 2: $b^2 - 4ac = 0$ (2 real repeated/double roots i.e. same/equal roots)
Case 3: $b^2 - 4ac < 0$ (no real roots)
Case 4: $b^2 - 4ac \geq 0$ (2 real roots i.e. doesn't mention whether distinct or repeated)

We are told 2 real equal roots hence we are in case 2

$$(10-p)^2 - 4(p)\left(\frac{5}{4}p - 5\right) > 0$$

Step 4: Now we have an inequality to solve to find p

$$(10-p)^2 - 4p\left(\frac{5}{4}p - 5\right) = 0$$
$$100 - 20p + p^2 - 5p^2 + 20p = 0$$
$$-4p^2 + 100 = 0$$
$$p^2 = 25$$
$$p = \pm 5$$

3.2.2 Inequalities

43)

$$kx^2 - (k-3)x + (k-8) = 0$$

When the questions talks about "number of roots" or "number of solutions" we must use the discriminant. We are finding the actual solution, we are just talking about "how many" and this is our hint to use the solution.

Step 1: Make sure we have 0 on one side. We already have this.

Step 2: locate $a, b,$and c

$$kx^2 - (k-3)x + (k-8) = 0$$
a =number in front of $x^2 = k$
b =number in front of $x = -k + 3$
c =number without an x or $x^2 = k - 8$

Note: Don't let the fact that we have a k confuse you. k is considered a constant here. x is the variable of the equation and that is the non-constant i.e. the variable that we don't include in a, b or c

$$kx^2 - (k-3)x + (k-8) = 0$$

Step 3: We are given what case we are in, so we plug into the correct formula for that case

Case 1: $b^2 - 4ac > 0$ (2 real distinct/different roots)
Case 2: $b^2 - 4ac = 0$ (2 real repeated/double roots i.e. same/equal roots)
Case 3: $b^2 - 4ac < 0$ (no real roots)
Case 4: $b^2 - 4ac \geq 0$ (2 real roots i.e. doesn't mention whether distinct or repeated)

We are told no real roots hence we are in case 3

$$(3-k)^2 - 4(k)(k-8) < 0$$

Step 4: Now we have an inequality to solve to find p

$$(3-k)^2 - 4(k)(k-8) < 0$$
$$9 + k^2 - 6k - 4k^2 + 32k < 0$$
$$-3k^2 + 26k + 9 < 0$$
$$3k^2 - 26k - 9 < 0$$
$$(k-9)(3k+1) < 0$$

$$k = 9, k = -\frac{1}{3}$$

Watch out for a really common mistake: Do not just guess the signs when solving a quadratic inequality, make sure to either graph or use a sign change test with a number line

$$k < -\frac{1}{3}, k > 9$$

44)

$$(k+3)x^2 + 6x + k = 5$$

When the questions talks about "number of roots" or "number of solutions" we must use the discriminant. We are finding the actual solution, we are just talking about "how many" and this is our hint to use the solution.

Step 1: Make sure we have 0 on one side.

$$(k+3)x^2 + 6x + k - 5 = 0$$

Step 2: locate a, b,and c

$$(k+3)x^2 + 6x + k - 5 = 0$$

a =number in front of $x^2 = k + 3$

b =number in front of $x = 6$

c =number without an x or $x^2 = k - 5$

Note: Don't let the fact that we have a k confuse you. k is considered a constant here. x is the variable of the equation and that is the non-constant i.e. the variable that we don't include in a, b or c

$$(k-3)x^2 + 6x + (k-5) = 0$$

Step 3: We are given what case we are in, so we plug into the correct formula for that case

Case 1: $b^2 - 4ac > 0$ (2 real distinct/different roots)

Case 2: $b^2 - 4ac = 0$ (2 real repeated/double roots i.e. same/equal roots)

Case 3: $b^2 - 4ac < 0$ (no real roots)

Case 4: $b^2 - 4ac \geq 0$ (2 real roots i.e. doesn't mention whether distinct or repeated)

We are told no real roots hence we are in case 3

$$6^2 - 4(k+3)(k-5) > 0$$

Step 4: Now we have an inequality to solve to find k

$$6^2 - 4(k+3)(k-5) > 0$$
$$36 - 4(k^2 - 2k - 15) > 0$$
$$36 - 4k^2 + 8k + 60 > 0$$
$$-4k^2 + 8k + 96 > 0$$
$$k^2 - 2k - 24 < 0$$
$$(k-6)(k+4) < 0$$
$$k = 6, k = -4$$

Watch out for a really common mistake: Do not just guess the signs when solving a quadratic inequality, make sure to either graph or use a sign change test with a number line

$$-4 < k < 6$$

45)

i.

$$(2k-7)x^2 - (k-2)x + (k-3) = 0$$

When the questions talks about "number of roots" or "number of solutions" we must use the discriminant. We are finding the actual solution, we are just talking about "how many" and this is our hint to use the solution.

Step 1: Make sure we have 0 on one side. We already have this.

Step 2: locate $a, b,$and c

$$(2k-7)x^2-(k-2)x+k-3=0$$
$$a = \text{number in front of } x^2 = 2k-7$$
$$b = \text{number in front of } x = 2-k$$
$$c = \text{number without an } x \text{ or } x^2 = k-3$$

Note: Don't let the fact that we have a k confuse you. k is considered a constant here. x is the variable of the equation and that is the non-constant i.e. the variable that we don't include in a, b or c

$$(2k-7)x^2-(k-2)x+(k-3)=0$$

Step 3: We are given what case we are in, so we plug into the correct formula for that case

Case 1: $b^2-4ac>0$ (2 real distinct/different roots)
Case 2: $b^2-4ac=0$ (2 real repeated/double roots i.e. same/equal roots)
Case 3: $b^2-4ac<0$ (no real roots)
Case 4: $b^2-4ac\geq 0$ (2 real roots i.e. doesn't mention whether distinct or repeated)

We are told real roots, doesn't mention whether distinct or repeated, hence we are in case 4:

$$(2-k)^2-4(2k-7)(k-3)\geq 0$$
$$(-k+2)^2-4(2k^2-13k+21)\geq 0$$
$$k^2-4k+4-8k^2+52k-84\geq 0$$
$$-7k^2+48k-80\geq 0$$
$$7k^2-48k+80\leq 0$$

ii.

Now we have an inequality to solve to find k

$$7k^2-48k+80\leq 0$$
$$(k-4)(7k-20)\leq 0$$
$$k=4, k=\frac{20}{7}$$

Watch out for a really common mistake: Do not just guess the signs when solving a quadratic inequality, make sure to either graph or use a sign change test with a number line

$$\frac{20}{7}\leq k\leq 4$$

46)

i.

$$k(3x^2+8x+9)=2-6x$$

When the questions talks about "number of roots" or "number of solutions" we must use the discriminant. We are finding the actual solution, we are just talking about "how many" and this is our hint to use the solution.

Step 1: Make sure we have 0 on one side.

$$3kx^2+8kx+9k=2-6x$$
$$3kx^2+(8k+6)x+(9k-2)=0$$

Step 2: locate $a, b,$and c

$$3kx^2+(8k+6)x+(9k-2)=0$$
$$a = \text{number in front of } x^2 = 3k$$
$$b = \text{number in front of } x = 8k+6$$
$$c = \text{number without an } x \text{ or } x^2 = 9k-2$$

Note: Don't let the fact that we have a k confuse you. k is considered a constant here. x is the variable of the equation and that is the non-constant i.e. the variable that we don't include in a, b or c

$$3kx^2-(8k+6)x+(9k-2)=0$$

Step 3: We are given what case we are in, so we plug into the correct formula for that case

Case 1: $b^2 - 4ac > 0$ (2 real distinct/different roots)

Case 2: $b^2 - 4ac = 0$ (2 real repeated/double roots i.e. same/equal roots)

Case 3: $b^2 - 4ac < 0$ (no real roots)

Case 4: $b^2 - 4ac \geq 0$ (2 real roots i.e. doesn't mention whether distinct or repeated)

We are told no real roots, hence we are in case 3:

$$(8k + 6)^2 - 4(3k)(9k - 2) < 0$$
$$64k^2 + 96k + 36 - 108k^2 + 24k < 0$$
$$-44k^2 + 120k + 36 < 0$$
$$11k^2 - 30k - 9 > 0$$

ii.

Now we have an inequality to solve to find k

$$11k^2 - 30k - 9 > 0$$
$$(11k + 3)(k - 3) > 0$$
$$k = -\frac{3}{11}, k = 3$$

Watch out for a really common mistake: Do not just guess the signs when solving a quadratic inequality, make sure to either graph or use a sign change test with a number line

$$k < -\frac{3}{11}, k > 3$$

3.3 Given number of roots, showing existence for all real values

47)

We are not solving for a specific value of k here. We want to show true for all values of k, so we will need to show that the function is always positive, regardless of the value of k

$b^2 - 4ac > 0$ when we have two distinct real solutions. We can't use this equation to solve for k this time. We must show always true.

$$1x^2 - (5 - k)x - (k + 2) = 0$$

a =number in front of $x^2 = 1$

b =number in front of $x = -5 + k$

c =number without an x or $x^2 = -k - 2$

$$b^2 - 4ac = (-5 + k)^2 - 4(1)(-k - 2)$$
$$= 25 - 10k + k^2 + 4k + 8$$
$$= k^2 - 6k + 33$$

To show this is always positive we must complete the square

$$(k - 3)^2 + 33 - 9$$
$$(k - 3)^2 + 24$$

$(k - 3)^2 > 0$ always and $24 > 0$

The sum of 2 positive numbers is always positive

$\therefore$ the discriminant is always greater than zero, so 2 distinct roots for any values of k

48)

$$f(x) = x^2 + (k + 3)x + k$$

i.

Let $f(x) = 0$.

$$1x^2 + (k + 3)x + k = 0$$

a =number in front of $x^2 = 1$

b =number in front of $x = k + 3$

c =number without an x or $x^2 = k$

Discriminant:

$$b^2 - 4ac = (k+3)^2 - 4(1)(k)$$

$$= k^2 + 6k + 9 - 4k = k^2 + 2k + 9$$

Complete the square,

$$k^2 + 2k + 9 = (k+1)^2 + 8$$

ii.

To show that the equation has real roots, $b^2 - 4ac \geq 0$.

$$(k+1)^2 + 8 \geq 0$$

$(k+1)^2$ is always positive since the square of a number is always positive.
8 is positive as well, so no matter what value k is, the result will always be greater than 0.
The sum of 2 positive numbers is always positive
$\therefore$ the discriminant is always greater than zero, so 2 distinct roots for any values of k

49)

$$h(x) = 2x^2 + (k+4)x + k$$

Let $h(x) = 0$.

$$2x^2 + (k+4)x + k = 0$$

a =number in front of $x^2 = 2$
b =number in front of $x = k + 4$
c =number without an x or $x^2 = k$

Discriminant:

$$b^2 - 4ac = (k+4)^2 - 4(2)(k)$$

$$= k^2 + 8k + 16 - 8k = k^2 + 16$$

k^2 will always be positive and when it is added with 16, the result will always be greater than 0.
Hence, it is proven that the $h(x)$ has two distinct real roots for all values of k.

50)

$$(q-5)x^2 + 5x - q = 0$$

a =number in front of $x^2 = q - 5$
b =number in front of $x = 5$
c =number without an x or $x^2 = -q$

Discriminant:

$$b^2 - 4ac = (5)^2 - 4(q-5)(-q)$$

$$= 25 + 4q^2 - 20q$$
$$= 25 + 4(q^2 - 5q)$$

Complete the square.

$$= 25 + 4\left[\left(q - \frac{5}{2}\right)^2 - \frac{25}{4}\right]$$
$$= 25 + 4\left(q - \frac{5}{2}\right)^2 - 25$$
$$= 4\left(q - \frac{5}{2}\right)^2$$

$\left(q - \frac{5}{2}\right)^2$ will always be positive since it is a square and when multiply by 4, the result is positive as well.
Hence, it is proven that the equation has real roots for any values of q.

51)

i.

$$f(x) = 4kx^2 + (4k+2)x + 1$$

Let $f(x) = 0$.

$$4kx^2 + (4k+2)x + 1 = 0$$

a =number in front of $x^2 = 4k$

b =number in front of $x = 4k + 2$

c =number without an x or $x^2 = 1$

Discriminant:

$$b^2 - 4ac = (4k+2)^2 - 4(4k)(1)$$

$$= 16k^2 + 16k + 4 - 16k$$

$$= 16k^2 + 4$$

$16k^2$ is always greater than 0 and the result will always be positive when $16k^2$ is added with 4, so the discriminant of $f(x) > 0$ for all non-zero values of k.

ii.

When $k = 0$,

$$f(x) = 4(0)x^2 + (4(0) + 2)x + 1$$
$$= 2x + 1$$

It is a linear function with only one root.

3.4 Hidden Discriminant

3.4.1 Tangent to a circle

52)

$$x^2 + y^2 + 4x - 10y - 7 = 0,\ y = 2x + c$$

i.

Substitute $y = 2x + c$ into $x^2 + y^2 + 4x - 10y - 7 = 0,$

$$x^2 + (2x+c)^2 + 4x - 10(2x+c) - 7 = 0$$
$$x^2 + 4x^2 + 4cx + c^2 + 4x - 20x - 10c - 7 = 0$$

We have multiple terms containing x so we collect them.

$$5x^2 + (4c - 16)x + c^2 - 10c - 7 = 0$$
$$5x^2 + 4(c-4)x + (c^2 - 10c - 7) = 0$$

ii.

Tangent $\Longrightarrow b^2 - 4ac = 0$

$$5x^2 + 4(c-4)x + (c^2 - 10c - 7) = 0$$

a =number in front of $x^2 = 5$

b =number in front of $x = 4(c-4)$

c =number without an x or $x^2 = c^2 - 10c - 7$

$$b^2 - 4ac = 0$$

$$[4(c-4)]^2 - 4(5)(c^2 - 10c - 7) = 0$$
$$16(c^2 - 8c + 16) - 20(c^2 - 10c - 7) = 0$$
$$4(c^2 - 8c + 16) - 5(c^2 - 10c - 7) = 0$$
$$4c^2 - 32c + 64 - 5c^2 + 50c + 35 = 0$$
$$-c^2 + 18c + 99 = 0$$
$$c^2 - 18c - 99 = 0$$

iii.

$$1c^2 - 18c - 99 = 0$$

Solve using quadratic formula

a =number in front of $c^2 = 1$
b =number in front of $c = -18$
c =number without an c or $c^2 = -99$

$$c = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$c = \frac{-(-18) \pm \sqrt{(-18)^2 - 4(1)(-99)}}{2(1)}$$

$$= \frac{18 \pm \sqrt{720}}{2} = \frac{18 \pm 12\sqrt{5}}{2}$$

$$c = 9 \pm 6\sqrt{5}$$

53)

$$mx - y - 2 = 0, x^2 + 6x + y^2 - 8y = 4$$

From $mx - y - 2 = 0$,

$$y = mx - 2$$

Substitute $y = mx - 2$ into $x^2 + 6x + y^2 - 8y = 4$,

$$x^2 + 6x + (mx - 2)^2 - 8(mx - 2) = 4$$

$$x^2 + 6x + m^2x^2 - 4mx + 4 - 8mx + 16 = 4$$

We have multiple terms containing x and x^2 so we collect them.

$$(1 + m^2)x^2 + (6 - 12m)x + 16 = 0$$

The line touches the circle $\Longrightarrow b^2 - 4ac = 0$

$$(1 + m^2)x^2 + (6 - 12m)x + 16 = 0$$

a =number in front of $x^2 = 1 + m^2$
b =number in front of $x = (6 - 12m)$
c =number without an x or $x^2 = 16$

$$b^2 - 4ac = 0$$

$$(6 - 12m)^2 - 4(1 + m^2)(16) = 0$$

$$36 - 144m + 144m^2 - 64 - 64m^2 = 0$$

$$80m^2 - 144m - 28 = 0$$

$$20m^2 - 36m - 7 = 0$$

Solve using quadratic formula

a =number in front of $m^2 = 20$
b =number in front of $m = -36$
c =number without an m or $m^2 = -7$

$$c = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$c = \frac{-(-36) \pm \sqrt{(-36)^2 - 4(20)(-7)}}{2(20)}$$

$$\frac{36 \pm \sqrt{1856}}{40} = \frac{36 \pm 8\sqrt{29}}{40}$$

$$m = \frac{9 \pm 2\sqrt{29}}{10}$$

54)

$$2x + y - 5 = 0, (x - 3)^2 + (y - p)^2 = 5$$

i.

From $2x + y - 5 = 0$,

$$y = 5 - 2x$$

Substitute $y = 5 - 2x$ into $(x-3)^2 + (y-p)^2 = 5$,

$$(x-3)^2 + (5 - 2x - p)^2 = 5$$
$$x^2 - 6x + 9 + 25 - 10x - 5p - 10x + 4x^2 + 2px - 5p + 2px + p^2 = 5$$
$$5x^2 - 26x + 4px + 29 - 10p + p^2 = 0$$

We have multiple terms containing x so we collect them.

$$5x^2 + (4p - 26)x + (p^2 - 10p + 29) = 0$$

Tangent $\Longrightarrow b^2 - 4ac = 0$

$$5x^2 + (4p - 26)x + (p^2 - 10p + 29) = 0$$
$$b^2 - 4ac = 0$$

$$(4p - 26)^2 - 4(5)(p^2 - 10p + 29) = 0$$
$$16p^2 - 208p + 676 - 20p^2 + 200p - 580 = 0$$
$$-4p^2 - 8p + 96 = 0$$
$$p^2 + 2p - 24 = 0$$
$$(p-4)(p+6) = 0$$
$$p = 4, p = -6$$

ii.

When $p = 4$,

$$(x-3)^2 + (y-4)^2 = 5$$

$\therefore$ Centre of circle $= (3, 4)$

When $p = -6$,

$$(x-3)^2 + (y - -6)^2 = 5$$

$\therefore$ Centre of circle $= (3, -6)$

55)

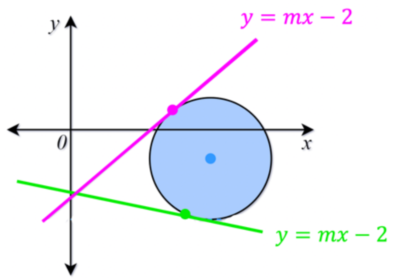
A line passing through $(0, -2)$ has y intercept -2 so therefore can be generally written as $\boldsymbol{y = mx - 2}$.

Intersects so lets represent this equation by solving $(x-7)^2 + (y+1)^2 = 5$ and $y = mx - 2$ simultaneously

$$(x-7)^2 + ((mx-2)+1)^2 = 5$$
$$(x-7)^2 + (mx-1)^2 = 5$$
$$x^2 - 14x + 49 + m^2x^2 - 2mx + 1 = 5$$
$$(1 + m^2)x^2 + (-14 - 2m)x + 45 = 0$$

Touches once so $b^2 - 4ac = 0$

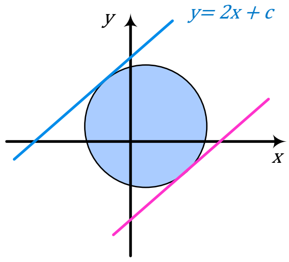
$$a = 1 + m^2$$
$$b = -14 - 2m$$
$$c = 45$$
$$(-14 - 2m)^2 - 4(1 + m^2)(45) = 0$$
$$196 + 56m + 4m^2 - 180 - 180m^2 = 0$$
$$176m^2 - 56m - 16 = 0$$
$$44m^2 - 14m - 9 = 0$$
$$22m^2 - 7m - 2 = 0$$
$$(2m - 1)(11m + 2) = 0$$
$$\boldsymbol{m = 0.5}, m = -\frac{2}{11}$$



Told positive gradient so choose $m = 0.5$

$$y = 0.5x - 2$$

56)



A line has gradient of 2 it can therefore be generally written as $y = 2x + c$.

Intersects so lets represent this equation by solving $(x+5)^2 + (y+3)^2 = 80$ and $y = 2x + c$ simultaneously

$$(x+5)^2 + ((2x+c)+3)^2 = 80$$
$$(x+5)^2 + (2x+c+3)^2 = 80$$
$$x^2 + 10x + 25 + 4x^2 + 2cx + 6x + 2cx + c^2 + 3c + 6x + 3c + 9 = 80$$
$$5x^2 + 22x + 4cx - 46 + c^2 + 6c = 0$$
$$5x^2 + (4c+22)x + (c^2 + 6c - 46) = 0$$

Tangent so $b^2 - 4ac = 0$

$$a = 5$$
$$b = 4c + 22$$
$$c = c^2 + 6c - 46$$
$$(4c+22)^2 - 4(5)(c^2 + 6c - 46) = 0$$
$$16c^2 + 176c + 484 - 20c^2 - 120c + 920 = 0$$
$$-4c^2 + 56c + 1404 = 0$$
$$c^2 - 14c - 351 = 0$$
$$(c-27)(c+13) = 0$$
$$c = 27, c = -13$$

When $c = 27$,

$$y = 2x + 27$$

When $c = -13$,

$$y = 2x - 13$$

∴ The two possible equations for l are $y = 2x + 27$ and $y = 2x - 13$.

57)

The line has gradient of -3 it can therefore be generally written as $y = -3x + c$.

Intersects so lets represent this equation by solving $(x-5)^2 + (y+3)^2 = 10$ and $y = -3x + c$ simultaneously

$$(x-5)^2 + ((-3x+c)+3)^2 = 10$$
$$(x-5)^2 + (-3x+c+3)^2 = 10$$
$$x^2 - 10x + 25 + 9x^2 - 3cx - 9x - 3cx + c^2 + 3c - 9x + 3c + 9 = 10$$
$$10x^2 - 28x - 6cx + 24 + c^2 + 6c = 0$$
$$10x^2 + (-28 - 6c)x + (c^2 + 6c + 24) = 0$$

Tangent so $b^2 - 4ac = 0$

$$a = 10$$
$$b = -28 - 6c$$
$$c = c^2 + 6c + 24$$
$$(-28 - 6c)^2 - 4(10)(c^2 + 6c + 24) = 0$$
$$784 + 36c^2 + 336c - 40c^2 - 240c - 960 = 0$$
$$-4c^2 + 96c - 176 = 0$$
$$c^2 - 24c + 44 = 0$$
$$(c-22)(c-2) = 0$$
$$c = 22, c = 2$$

When $c = 22$,

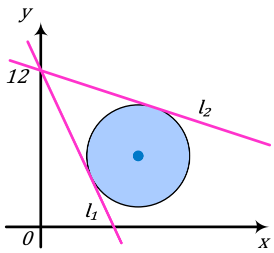
$$y = -3x + 22$$

When $c = 2$,

$$y = -3x + 2$$

∴ The two possible equations of l are $y = -3x + 22$ and $y = -3x + 2$.

58)



Since both the lines cross the y axis at $y = 12$, they both have the same general equation of $y = mx + 12$.

Intersects so lets represent this equation by solving $(x-6)^2 + (y-5)^2 = 17$ and $y = mx + 12$ simultaneously

$$(x-6)^2 + (mx + 12 - 5)^2 = 17$$
$$(x-6)^2 + (mx+7)^2 = 17$$
$$x^2 - 12x + 36 + m^2x^2 + 14mx + 49 = 17$$
$$(1+m^2)x^2 + (14m - 12)x + 68 = 0$$

Touches once so $b^2 - 4ac = 0$

$$a = 1 + m^2$$
$$b = 14m - 12$$
$$c = 68$$
$$(14m-12)^2 - 4(1+m^2)(68) = 0$$
$$196m^2 - 336m + 144 - 272m^2 - 272 = 0$$
$$-76m^2 - 336m - 128 = 0$$
$$19m^2 + 84m + 32 = 0$$
$$(19m + 8)(m + 4) = 0$$
$$m = -\frac{8}{19}, m = -4$$

Since l_1 is steeper, it has the equation of $y = -4x + 12$ while l_2 has the equation of $y = -\frac{8}{19}x + 12$.

3.4.2 Showing always positive or negative

59)

Let's re-arrange first

$x^2 + 6.5x + 16 > 0$

We need to show this is always positive

Way 1: Complete the square	Way 2: Use discriminant
Let's complete the square to do this $x^2 + 6.5x + 16 = (x + 3.25)^2 + 16 - 10.5625$ $= (x+3.25)^2 + 5.4375$ $(x+3.25)^2 > 0$ always and $5.4375 > 0$ The sum of 2 positive numbers is always positive ∴ always greater than zero so $x^2 + 6.5x + 16 > 0$ Hence $x^2 + 6x + 18 > 2 - \frac{1}{2}x$	$1x^2 + 6.5x + 16 > 0$ $b^2 - 4ac = 6.5^2 - 4(1)(16) = -21.75$ $-21.75 < 0$ This means no real roots, so doesn't cross x axis We also have a $+x^2$ term so happy face (positive parabola) Therefore always above x axis $x^2 + 6.5x + 16 > 0$ Hence $x^2 + 6x + 18 > 2 - \frac{1}{2}x$

60)

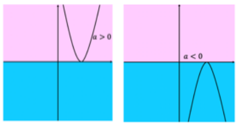
Way 1: Discriminant:

$1x^2 - 2x + 1 > 0$

$b^2 - 4ac = (-2)^2 - 4(1)(1) = 4 - 4 = 0$

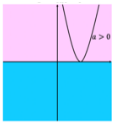
so there is only one real root (repeated root), meaning the curve touches the x axis only once.

We are in one of these 2 cases



The positive coefficient of the x^2 term means that the graph is always above or on the x axis i.e. in the positive y region or on the x axis

So, we must be in the case



and we know that $f(x) = y$

hence it is obvious that $f(x) \geq 0$ for all real values of x.

Way 2: Complete The Square

This is already in the right form

$(x - 1)^2$

$(x - 1)^2$ is always positive since it is squared

hence

$(x - 1)^2 \geq 0$

Therefore $(x - 1)^2$ is always positive regardless of the values of x

61)

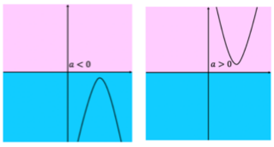
Way 1: Discriminant:

$1x^2 + 16 > 0$

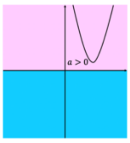
$b^2 - 4ac = 0^2 - 4(1)(16) = -64 < 0$

so there are no real roots (the curve doesn't cross the x axis)

We are in one of these 2 cases



The positive coefficient of the x^2 term means that the graph is always above the x axis i.e. in the positive y region, so we must be in the case



Hence it obvious that $f(x) > 0$ for all real values of x.

Way 2: Complete The Square

This is already in the right form

$x^2 + 16$

$x^2 > 0$ since squared and $+16 > 0$

so we have the sum of 2 positive terms, hence

$x^2 + 16 > 0$

Therefore $x^2 + 16$ is always positive regardless of the values of x

62)

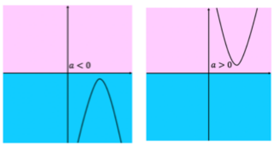
Way 1: Discriminant:

$1x^2 - 8x + 18 > 0$

$b^2 - 4ac = (-8)^2 - 4(1)(18) = 64 - 72 = -8 < 0$

so there are no real roots (the curve doesn't cross the x axis)

We are in one of these 2 cases



The positive coefficient of the x^2 term means that the graph is always above the x axis i.e. in the positive y region, so we must be in the case

Way 2: Complete The Square

$x^2 - 8x + 18 = (x - 4)^2 + 18 - 16$

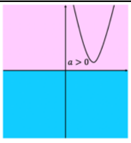
$= (x - 4)^2 + 2$

$(x - 4)^2 > 0$ since squared and $+2 > 0$

so we have the sum of 2 positive terms, hence

$x^2 - 8x + 18 > 0$

Therefore $x^2 - 8x + 18$ is always positive regardless of the values of x



Hence it obvious that $f(x) > 0$ for all real values of x.

63)

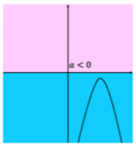
Way 1: Discriminant:

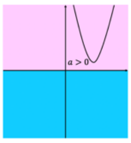
$2x^2 + 8x + 9$

$b^2 - 4ac = 8^2 - 4(2)(9) = 64 - 72 = -8 < 0$

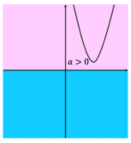
so there are no real roots (the curve doesn't cross the x axis)

We are in one of these 2 cases





The positive coefficient of the x^2 term means that the graph is always above the x axis i.e. in the positive y region, so we must be in the case



∴ The function is always positive for any value of x.

Way 2: Complete The Square

$$2x^2 + 8x + 9 = 2\left(x^2 + 4x + \frac{9}{2}\right)$$

$$= 2\left[(x+2)^2 + \frac{9}{2} - 4\right]$$

$$= 2(x+2)^2 + 1$$

$2 > 0, (x+2)^2 > 0$ since squared, and $+1 > 0$ so we have the sum of 2 positive terms, hence

$2x^2 + 8x + 9 > 0$

Therefore $2x^2 + 8x + 9$ is always positive regardless of the values of x

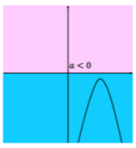
64)

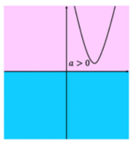
$$f(x) = -5x^2 - 1x - 10$$

Discriminant:

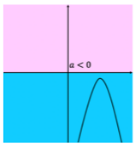
$$b^2 - 4ac = (-1)^2 - 4(-5)(-10)$$
$$= 1 - 200 = -199$$
$$-199 < 0$$

There are no real roots, the curve doesn't cross the x axis.





The graph is negative by looking at x^2 term, hence it is always below x axis.



∴ $f(x) < 0$ for all real values of x.

4 Diamond

4.1 Given number of roots, solve for an unknown

4.1.1 Equalities

65)

$$2kx^2 - 4kx + 1 = 0$$

i. As there's to equal real solutions (one repeated root),

$$b^2 - 4ac = 0$$
$$(-4k)^2 - 4(2k)(1) = 0$$
$$16k^2 - 8k = 0$$
$$8k(2k - 1) = 0$$
$$k = 0, k = \frac{1}{2}$$

Since $k \neq 0$, $k = \frac{1}{2}$.

ii. Substitute $k = \frac{1}{2}$ into $2kx^2 - 4kx + 1 = 0$,

$$2\left(\frac{1}{2}\right)x^2 - 4\left(\frac{1}{2}\right)x + 1 = 0$$
$$x^2 - 2x + 1 = 0$$

To find the vertex, $\frac{dy}{dx} = 0$

$$\frac{dy}{dx} = 2x - 2$$
$$2x - 2 = 0$$
$$2x = 2$$
$$x = 1$$

The vertex of the graph is $(1, 0)$. It is on the x axis, so the horizontal line of $y = p$ must be on or above x axis in order to intersect the curve.

$$\therefore p \geq 0$$

4.1.2 Inequalities

66)

$$mx^2 - 2(m + 2)x + m + 2 = 0$$

i. As there's to equal real solutions (one repeated root),

$$b^2 - 4ac = 0$$
$$\left(-2(m+2)\right)^2 - 4(m)(m+2) = 0$$
$$(-2m-4)^2 - 4m^2 - 8m > 0$$
$$4m^2 + 16m + 16 - 4m^2 - 8m > 0$$
$$8m + 16 > 0$$
$$8m > -16$$
$$m > -2$$

ii.

Hint: to have opposite signs i.e one positive and one negative, the product of their roots must be negative i.e $\frac{m+2}{m} < 0$.
The form of a quadratic is $x^2 - (sum\ roots)x + (product\ roots)$. Divide all terms by coefficient in front of x^2 to get the question into this form.

$$mx^2 - 2(m+2)x + m + 2 = 0$$

Divide by m.

$$x^2 - \frac{2(m+2)}{m}x + \frac{m+2}{m} = 0$$

Product of roots must be negative.

$$\frac{m+2}{m} < 0$$

Denominator must not be zero, so $m \neq 0$.

We need to draw a number line and perform a sign test to solve an inequality (or graph instead)
We have three possibilities for this solution, they are $m < -2$, $m > 0$ and $-2 < m < 0$.
Let's test these one-by-one.

When $m < -2$,
Assume $m = -3$.

$$\frac{-3+2}{-3} = \frac{1}{3} > 0$$

So this is shown that $m < -2$ is invalid.

Let's now test $m > 0$.
Assume $m = 1$.

$$\frac{1+2}{1} = 3 > 0$$

The result is also greater than 0, which means $m > 0$ is also invalid.

We are only left with $-2 < m < 0$, and obviously this is the answer after eliminating the other two, but let's test it.
Assume $m = -1$.

$$\frac{-1+2}{-1} = -1 < 0$$

∴ The range of values of m is $-2 < m < 0$

67)

$$f(x) = 2x^2 + (4k+3)x + (2k-1)(k+2) = 0$$

i. Let $f(x) = 0$

$$b^2 - 4ac = (4k+3)^2 - 4(2)(2k-1)(k+2)$$
$$= (4k+3)^2 - 4(2)(2k^2 + 3k - 2)$$
$$= 16k^2 + 24k + 9 - 16k^2 - 24k + 16$$
$$= 25$$

ii.

Hint: Use quadratic formula to find solutions and then put back into equation.

$$2x^2 + (4k+3)x + (2k-1)(k+2) = 0$$

When $f(x) = 0$,

$$x = \frac{-(4k+3) \pm \sqrt{(4k+3)^2 - 4(2)(2k^2+3k-2)}}{2(2)} = \frac{(-4k-3) \pm \sqrt{25}}{4} = \frac{-4k-3 \pm 5}{4}$$

$$\therefore x = -k + \frac{1}{2}, x = -k - 2$$

$$\therefore f(x) = \left(x + k - \frac{1}{2}\right)(x + k + 2)$$

4.2 Given number of roots, Showing existence for all real values

68)

$$\left(x+\frac{1}{2}\right)\left(x-\frac{3}{4}\right) = x^2 - \frac{3}{4}x + \frac{1}{2}x - \frac{3}{8}$$

As p, q and r are integers we must get rid of the fractions.

$$x^2 - \frac{3}{4}x + \frac{1}{2}x - \frac{3}{8} = 8x^2 - 6x + 4x - 3$$
$$= 8x^2 - 2x - 3$$

Compare coefficients.

$$px^2 + qx + r = 8x^2 - 2x - 3$$

$$p = 8$$
$$q = -2$$
$$r = -3$$

69)

i.

$$3x^2 + 2kx + k - 1 = 0$$

2 distinct roots, so $b^2 - 4ac > 0$:

$$(2k)^2 - 4(3)(k-1) > 0$$
$$4k^2 - 12k + 12 > 0$$
$$k^2 - 3k + 12 > 0$$
$$(k-1.5)^2 + 12 - 2.25 > 0$$
$$(k-1.5)^2 + 9.75 > 0$$

$(k-1.5)^2$, is squared so is always positive, 9.75 is always positive. You get a positive result when adding two positive terms together, therefore $3x^2 + 2kx + k - 1 = 0$ has two distinct real roots for all values of $k \in \mathcal{R}$.

ii.

Hint: Find the minimum value of the discriminant since roots are closest together when discriminant is least by differentiating and setting equal to 0

$$y = k^2 - 3k + 12$$
$$\frac{dy}{dx} = 2k - 3$$

Minimum when $\frac{dy}{dx} = 0$,

$$2k - 3 = 0$$
$$2k = 3$$
$$k = \frac{3}{2}$$

4.3 Hidden Discriminant - Combined With Other Topics

4.3.1 Showing Always Positive Or Always Negative

70)

$$f'(x) = x^2 - 8x + 18$$

Increasing when $1x^2 - 8x + 18 > 0$

Discriminant:

$$b^2 - 4ac = (-8)^2 - 4(1)(18) = 64 - 72 = -8 < 0$$

There are no real roots, the curve doesn't cross the x axis.

The graph is positive by looking at x^2 term, hence it is always above x axis.

Hence $f'(x) > 0$ always

∴ The function $f(x)$ is always increasing.

71)

Hint: solve discriminant of derivative ≤ 0

$$f(x) = px^3 + px^2 + qx$$

$$f'(x) = 3px^2 + 2px + q$$

Since $f'(x)$ is positive and $f'(x) \geq 0$, $b^2 - 4ac \leq 0$ so that the curve doesn't cross the x axis.

$$(2p)^2 - 4(3p)(q) \leq 0$$
$$4p^2 - 12pq \leq 0$$
$$p^2 - 3pq \leq 0$$
$$p^2 \leq 3pq$$

72)

Hint: If we rearrange this is like saying the function is always positive so then solve discriminant ≤ 0

$$m(x+1) \leq x^2$$
$$mx + m \leq x^2$$
$$1x^2 - mx - m \geq 0$$

The function is increasing and so $b^2 - 4ac \leq 0$.

$$(-m)^2 - 4(1)(-m) \leq 0$$
$$m^2 + 4m \leq 0$$
$$m(m+4) \leq 0$$
$$m = 0, m = -4$$

$$-4 \leq m \leq 0$$

73)

Hint: Solve discriminant less than zero and then solve term in front of x^2 to be positive.

$$f(x) = (2p-3)x^2 + 2x + p - 1$$
$$f(x) > 0$$

The function is positive and increasing, so $b^2 - 4ac < 0$.

$$2^2 - 4(2p-3)(p-1) < 0$$
$$4 - 4(2p^2 - 5p + 3) < 0$$
$$1 - 2p^2 + 5p - 3 < 0$$
$$2p^2 - 5p + 2 > 0$$
$$(p-2)(2p-1) > 0$$

$$p = 2, p = \frac{1}{2}$$
$$p < \frac{1}{2}, p > 2$$

Since the function is positive, $2p - 3 > 0$. For this to happen, $p > 2$.

4.3.2 Combined With Other Topics

4.3.2.1 Differentiation

74)

$$y = x^3 - 6x^2 + kx - 4$$
$$\frac{dy}{dx} = 3x^2 - 12x + k$$

Gradient is zero $\Longrightarrow \frac{dy}{dx} = 0$

$$3x^2 - 12x + k = 0$$

There is only one point at which the gradient is zero $\Longrightarrow b^2 - 4ac = 0$

$$(-12)^2 - 4(3)(k) = 0$$
$$144 - 12k = 0$$
$$12k = 144$$
$$k = 12$$

75)

Hint: plug both points into equation and solve simultaneously.

i.

$$y = 8x^3 + bx^2 + cx + d$$

Gradient is zero $\Longrightarrow \frac{dy}{dx} = 0$

$$\frac{dy}{dx} = 24x^2 + 2bx + c$$
$$24x^2 + 2bx + c = 0$$

2 distinct points $\Longrightarrow b^2 - 4ac > 0$

$$(2b)^2 - 4(24)(c) > 0$$
$$4b^2 - 96c > 0$$
$$4b^2 > 96c$$
$$b^2 > 24c$$

ii.

Substitute $x = \frac{1}{2}$ into $\frac{dy}{dx}$,

$$24\left(\frac{1}{2}\right)^2 + 2b\left(\frac{1}{2}\right) + c = 0$$
$$24\left(\frac{1}{4}\right) + b + c = 0$$
$$6 + b + c = 0$$
$$b = -6 - c$$

Substitute $x = -\frac{3}{2}$ into $\frac{dy}{dx}$,

$$24\left(-\frac{3}{2}\right)^2 + 2b\left(-\frac{3}{2}\right) + c = 0$$
$$24\left(\frac{9}{4}\right) - 3b + c = 0$$
$$54 - 3b + c = 0$$
$$c = 3b - 54$$

Substitute $b = -6 - c$ into $c = 3b - 54$,

$$c = 3(-6 - c) - 54$$
$$c = -18 - 3c - 54$$
$$4c = -72$$
$$c = -18$$
$$b = 12$$

Plug in the value of b and c into the equation of the curve.

$$\therefore y = 8x^3 + 12x^2 - 18x + d$$

Plug in $\left(\frac{1}{2}, -12\right)$,

$$-12 = 8\left(\frac{1}{8}\right) + 12\left(\frac{1}{4}\right) - 9 + d$$
$$-12 = 1 + 3 - 9 + d$$
$$d = -7$$

76)

$$y = \frac{x^2 - 1}{x + a}$$

Differentiate using quotient rule:

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$
$$\frac{dy}{dx} = \frac{(x+a)(2x) - (x^2-1)(1)}{(x+a)^2}$$
$$= \frac{2x^2 + 2ax - x^2 + 1}{(x+a)^2}$$
$$= \frac{x^2 + 2ax + 1}{(x+a)^2}$$

Turning points when $\frac{dy}{dx} = 0$.

$$\frac{x^2 + 2ax + 1}{(x+a)^2} = 0$$
$$1x^2 + 2ax + 1 = 0$$

No turning points $\Longrightarrow b^2 - 4ac < 0$

$$(2a)^2 - 4(1)(1) < 0$$
$$4a^2 - 4 < 0$$
$$a^2 - 1 < 0$$
$$a^2 < 1$$
$$-1 < a < 1$$

77)

$$f(x) = e^{mx}(x^2 + x)$$
$$f'(x) = e^{mx}(2x + 1) + (x^2 + x)(me^{mx})$$
$$= e^{mx}(2x + 1 + mx^2 + mx)$$
$$= e^{mx}[mx^2 + (m+2)x + 1]$$

To find the stationary points, equate $f'(x)$ to 0.
When $f'(x) = 0$,

$$e^{mx}[mx^2 + (m+2)x + 1] = 0$$
$$mx^2 + (m+2)x + 1 = 0$$

Find the discriminant.

$$b^2 - 4ac = (m+2)^2 - 4(m)(1)$$
$$= m^2 + 4m + 4 - 4m$$
$$= m^2 + 4 > 0$$

$\therefore f$ has two stationary points.

4.3.2.2 Logs

78)

$$f(x) = \log_k(6x - 3x^2)$$

When $f(x) = 2$,

$$\log_k(6x - 3x^2) = 2$$
$$6x - 3x^2 = k^2$$
$$3x^2 - 6x + k^2 = 0$$

Has exactly one solution $\Longrightarrow b^2 - 4ac = 0$

$$(-6)^2 - 4(3)(k^2) = 0$$
$$36 - 12k^2 = 0$$
$$12k^2 = 36$$
$$k^2 = 3$$
$$k = \pm\sqrt{3}$$

Since $k > 0$, $k = \sqrt{3}$.

4.3.2.3 Trig

79)

$f(x) = 3\tan^4 x + 2k$, $g(x) = -\tan^4 x + 8k\tan^2 x + k$

Equate both equations.

$$3\tan^4 x + 2k = -\tan^4 x + 8k\tan^2 x + k$$
$$4\tan^4 x - 8k\tan^2 x + k = 0$$

Let $\tan^2 x = y$.

$$4y^2 - 8ky + k = 0$$

Intersect at exactly one point $\Longrightarrow b^2 - 4ac = 0$

$$(-8k)^2 - 4(4)(k) = 0$$
$$64k^2 - 16k = 0$$
$$16k(4k-1) = 0$$
$$k = 0, k = \frac{1}{4}$$

Since $0 < k < 1$, $k = \frac{1}{4}$.

4.3.2.4 Quadratic Modelling

80)

$h(x) = -\frac{3}{10}x^2 + \frac{5}{2}x + \frac{3}{2}$, $h(x) = \frac{15}{2} - \frac{1}{5}x$

We first have to equate both equations.

$$-\frac{3}{10}x^2 + \frac{5}{2}x + \frac{3}{2} = \frac{15}{2} - \frac{1}{5}x$$
$$-3x^2 + 25x + 15 = 75 - 2x$$
$$3x^2 - 27x + 60 = 0$$
$$1x^2 - 9x + 20 = 0$$

Find the discriminant.

$$b^2 - 4ac = (-9)^2 - 4(1)(20)$$

$$= 81 - 80 = 1 > 0$$

∴ There are 2 real solutions, and hence the ball will hit the ceiling.